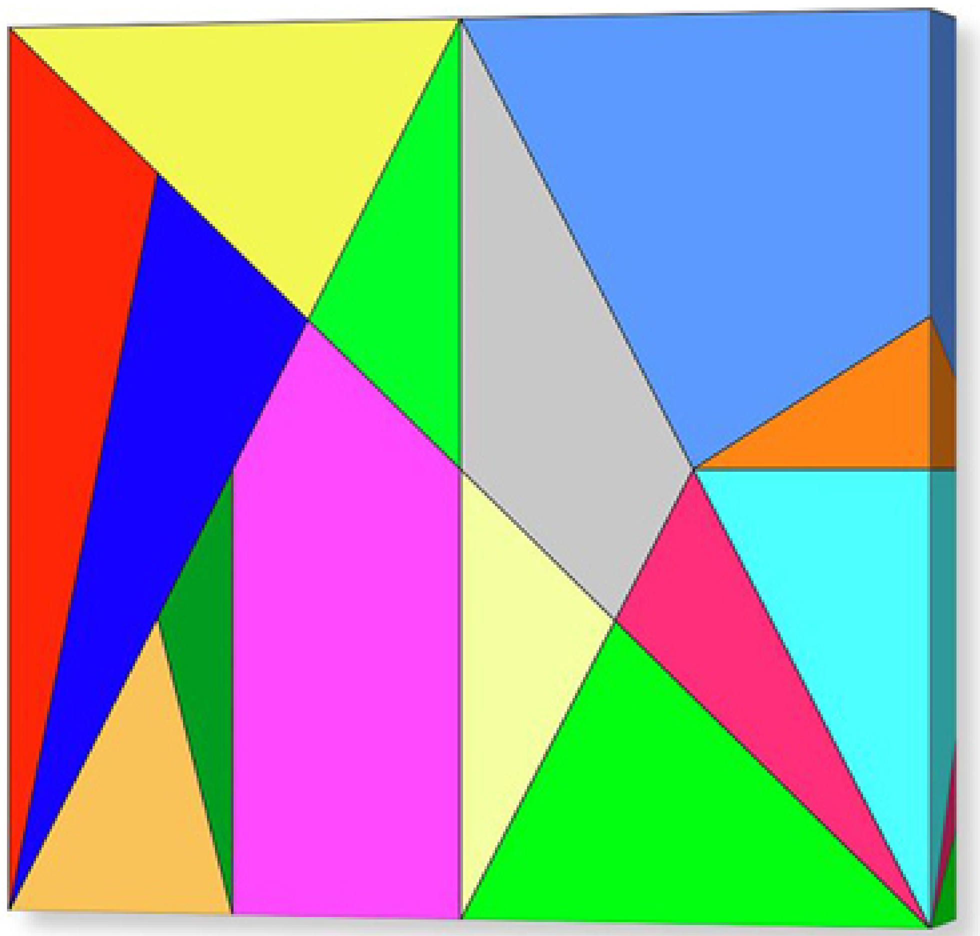
BY READING ARCHIMEDES

INVESTIGATIONS ON SOME THEOREMS AND CONJECTURES WITH HELP OF COMPUTER



Luciano Ancora

On the cover: Archimedes' ***Stomachion*** (in Greek "Στομάχιον"). The oldest known mathematical puzzle dates from Archimedes, more than two millennia ago. The game is a 14-piece dissection puzzle forming a square.

A Stomachion web page:
http://www.mcs.drexel.edu/~crorres/Archimedes/Stomachion/intro.html

SUMMARY

Introduction

You happened to find, in the introduction to some popular text of mathematics, a recommendation of the author to read bearing in hand pencil and paper, to be able to remake yourself calculations, drawings, reasoning. This will certainly achieves, through the study, a more secure knowledge of the facts of mathematics.

But you can do more. You can hold on, while you read, a personal computer, to be able to use, if necessary, its extraordinary computing power and graphics. So you can get results that go beyond the mere acquisition of knowledge.

A reader that has sufficient sense of observation and analytical skills, despite not having a high level of knowledge, may even make small "discoveries." You can in fact, using a spreadsheet, analyze complex calculations following their performance through the study of partial results, which often have interesting regularity or symmetry, sometimes indexes of recurring or inductive patterns which can lead to the discovery of general solutions of the problem under consideration. And again, using the CAD, you can take advantage of the computer graphics to build geometric models of the problem, and then observe them from different angles, looking for a solution process, which often shows itself in a surprising way.

Archimedes, in the introductory letter to his book "On the sphere and cylinder", referring to the results of his research on these objects, writes: *"... for these properties belong always to the nature of these figures it happened that many worthy of geometers front Eudoxus to ignore them all and no one understood them."* The properties of the objects of mathematics are then "preexisting" in the very nature of the objects, hidden from our thinking. The computer, amplifying our ability to investigate helps us to rediscover them, and they're finding is always a source of surprise and wonder.

The first three articles (at p. 7, 15 and 20) are in the format "publication" and together form a trilogy, having in common some elements of the number theory.

The first of them shows the *Square Pyramidal Number*, using which you can "*square up*" the parabolic segment. The demonstration contained in it has *heuristic* value, since there are no previous works that correlate the *Parabola* to the *Square Pyramidal Number*. It can be seen immediately on the Internet by typing together the *keywords* of these two mathematical objects.

The second article has originated from a search, auxiliary to the previous one, to achieve the well-known formula that calculates the sum of the squares of the first n natural numbers. At that time Internet was still at the origins and the need to know the formula, to develop the first work, led me to resort to "do it yourself". I then realized, much later, that the method used in those notes was new.

A third novelty is in the last article of the trilogy, born almost like a game in the wake of the second, where it is estimated that the British call: the *Squared Triangular Number*. This work is my favorite, for its originality and immediacy.

The collection continues with other items that do not offer anything new, while containing some a certain *didactic* value.

Luciano Ancora

Works of Archimedes

From: Opere di Archimede, Attilio Frajese (Utet, 1974).

1 On the equilibrium of planes
Work divided into two books in which it comes to concepts and propositions concerning the mechanics: statics in particular. In the first book treats the topic of the centers of gravity of the triangle and trapezoid. In the second book, according to the results of the first, it determines the center of gravity of the parabolic segment.

2 The quadrature of the parabola
The topic of this work is the quadrature of the parabolic segment, namely the construction (by ruler and compass) of a polygon equivalent to it. Archimedes solves the problem using first "mechanical" methods, and finally proves it geometrically by applying the rigorous "method of exhaustion".

3 On the sphere and the cylinder
Work in two books which can be considered the direct continuation of Euclid's Elements. The first book contains the fundamental results about the surface and the volume of the sphere. The second book consists of problems relating to the division of the sphere (using planes) into segments that satisfy certain conditions.

4 The method of mechanical theorems
Work lost through the centuries and happily rediscovered in 1906 by the Danish philologist Heiberg. In it is shown that "mechanical" method that Archimedes worked to find its results, which only later would prove geometrically in a rigorous way.

5 On spirals
In this work Archimedes introduces and studies that curve which was later called the spiral of Archimedes. The discussion proceeds on two topics, consisting in the use of the spiral for the rectification of the circle and in the quadrature of the areas between the spiral and certain lines.

6 On conoids and spheroids
Archimedes, in this original work, study some solids limited by "quadrics" surfaces. They are solids obtained by a complete rotation of a plane curve around a fixed axis. The work contains the basic results concerning the determination of volumes of the segments of cones and spheroid and its parts.

7 On floating bodies
This work is divided into two books, whose common argument is the immersion, total or partial, of a solid body into a liquid. In the first book is established and treated the famous principle of hydrostatic that posterity has rightly been called "Archimedes' principle".

8 On the measurement of a circle

This short work is probably an excerpt from another most complete work that has been lost. In it are contained only three propositions, which deal with the rectification of the circle and the quadrature of the circle. The third proposition, with its approximate determination of π, is one of the most well-known titles of merit of Archimedes.

9 The sand reckoner

In this work Archimedes poses the problem of counting the number of grains of sand in the sphere of the fixed stars, enclosing the universe. In doing so, devises an ingenious system for expressing very large numbers (extremely difficult for the Greeks, who did not have an efficient system of numbering). The work also contains the oldest evidence on the heliocentric system of Aristarchus of Samos and an interesting passage in which Archimedes, with simple means, measures the angle by which the solar disk is visible.

Quadrature of the parabola with the "square pyramidal number"

We perform here a new proof of the Archimedes theorem on the quadrature of the parabolic segment, executed without the aid of integral calculus, but using only the *square pyramidal number* and criteria for convergence of numerical sequences.
The translation of the discussion in the numerical field will happen using as unit of measurement of the areas involved in the proof, *equivalent triangles*, suitably identified in the grid of construction of the parabolic segment.

Introduction

The *Quadrature of the parabola* is one of the first works composed by Archimedes. It has as subject the quadrature of the parabolic segment, namely the construction (with ruler and compass) of a polygon equivalent to it. For *parabolic segment* Archimedes means the area between a straight line and a parabola, conceived as a section of a right cone. The work opens with an introduction to the basic properties of the parabola; then move to perform the quadrature of the parabola in a *mechanical way*, with considerations on the lever equilibrium; finally we get to the *geometric* proof of the quadrature, performed applying the *exhaustion method*.
Our proof revisits in a modern key the work of Archimedes, using the same figure that he uses in the proposition 16, where it is proved the fundamental result that the triangle ABC is triple of the parabolic segment. Archimedes uses a triangle ABC rectangle in B, having shown, in the previous proposition 15, that the result for such a situation generalizes to a parabolic segment with base non-perpendicular to the axis (note 1). In the next prop. 17 Archimedes infers from this result the other, more known, that the parabolic segment is 4/3 of the inscribed triangle.

Proposition 16

Let AB *the base of a parabolic segment, and draw through* B *the straight line* BC *parallel to the axis of the parabola to meet the tangent at* A *in* C. *I say that the area of the parabolic segment is one-third of the* ABC *triangle area.*

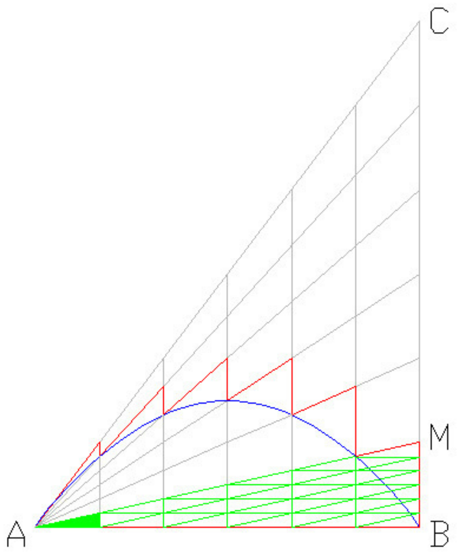


Fig. 1

Proof

Split the segments AB and BC into six equal parts and lead, from split points on AB parallels to BC, and from points on BC lines joining with A. The parabola passes through the points of intersection of the grid, as drawn, because, for one of its properties, it cuts the vertical lines of the grid in the same ratio in which the vertical lines cut the segment AB.
Consider the *saw-tooth figure* that circumscribes the parabolic segment. The area of this figure exceeds the area of the segment of a quantity that is equal to the overall area of the teeth. If we increase the number of divisions n su AB e BC, the excess area tends to zero as n tends to infinity. In other words: the area of the saw-tooth figure converges to the area of the parabolic segment, as n tends to infinity.
In the graph, the saw-tooth figure is divided into six vertical stripes composed: the first of 6 equivalent triangles, and the other strips, respectively, of 5, 4, 3, 2, 1 trapezoids, equivalent to each other in each strip. Now consider the triangle (shown in green) with a vertex at the point A. We will use this triangle as the measurement unit of the areas in the counts that follow:

The triangle ABM contains: 1+3+5+7+9+11 (sum of the first 6 odd numbers) = 6^2 green triangles (the sum of the first n odd numbers is n^2).

The triangle ABC contains: $6 \cdot 6^2 = 6^3$ green triangles.

In general, for any number n of divisions of AB and BC, *the triangle ABC contains* n^3 *green triangles.*

The circumscribed saw-tooth figure *A(cir.)* contains (for the equivalence of trapezes view above):

$$A(cir.) = 1\cdot 6 + 3\cdot 5 + 5\cdot 4 + 7\cdot 3 + 9\cdot 2 + 11\cdot 1 = 91 \quad \text{triangoli verdi.} \qquad (1)$$

We see that in (1) the succession of the addends is formed by products in which: the first factors give the sequence of the first six odd numbers and the second the succession, backward, of the first six natural numbers.

We represent now (1) with the following scheme (that also plays the dislocation of the trapezoids in the figure and their contents, making it more intelligible the count):

1	→					= 1	$= 1^2$
1	3	→				= 4	$= 2^2$
1	3	5	→			= 9	$= 3^2$
1	3	5	7	→		= 16	$= 4^2$
1	3	5	7	9	→	= 25	$= 5^2$
1	3	5	7	9	11	= 36	$= 6^2$
							91

Fig. 2

The schematized counting shows that the total number of green triangles in the *A(cir.)* figure is given by the sum of the squares of the first 6 natural numbers (there is also a diagonal path in the scheme, which leads to the same conclusion (note 5)).
It therefore appears that, the area of the saw-tooth figure, expressed in green triangles, is given by the *square pyramidal number P_6*:

$$A_{(cir.)} = P_6 = \sum_{k=1}^{6} k^2$$

The generalization to any number of divisions n, it follows from the possibility to extend the scheme of Figure 2 to the number n, adding rows containing successive sequences of odd numbers, until the n-th. The result is, in general, that:

$$A_n(cir.) = 1 \cdot n + 3 \cdot (n-1) + 5 \cdot (n-2) + ... + (2n-1) \cdot 1$$

And the area of the umpteenth saw-tooth figure, which circumscribes the parabolic segment, will be expressed by the *square pyramidal number P_n* :

$$A_{n\,(cir.)} = P_n = \sum_{k=1}^{n} k^2$$

This circumstance, together with the general result obtained for the area of triangle ABC (which is equal to n^3), we can reduce the proof to the simple check of the following relationship:

$$\lim_{n \to \infty} \frac{\sum_{k=1}^{n} k^2}{n^3} = \frac{1}{3} \qquad (2)$$

where the numerator to the first member is the nth square pyramidal number P_n.

You know that the sum in the numerator of (2) is: $P_n = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$

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after which the limit (2) follows from the fact that the ratio of two polynomials of the same degree in the variable n tends (as n tends to infinity) to the ratio between respective leading coefficients (coefficients of terms of maximum degree).

But (2) states that: the area (measured in green triangles) of the circumscribed figure is one-third the area of the triangle ABC, as n tends to infinity. Follows the statement in the proposition 16.

The proof "from below"

So that a proof could be called "complete" requires two estimates, one from *above* and one from *below*, ie, with a figure *out* and an *inside* from the parabolic segment.

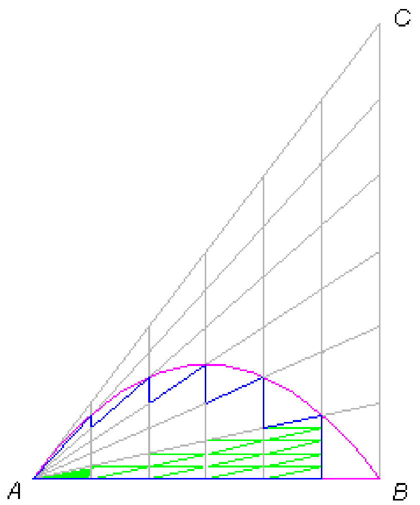


Fig. 3

Inscribing a saw-tooth figure *A(ins.)* in the parabolic segment, as in the figure, one can see that it contains:

$A(ins.) = 1 \cdot 5 + 3 \cdot 4 + 5 \cdot 3 + 7 \cdot 2 + 9 \cdot 1 = 55$ green triangles.

But the number 55 appears to be the 5th square pyramidal number; therefore, following the same reasoning made above, we can write:

$$A_n(ins.) = 1 \cdot (n-1) + 3 \cdot (n-2) + ... + (2n-3) \cdot 1$$

and, for the umpteenth area of the inscribed figure:

$$A_n(ins.) = P_{n-1} = \sum_{k=1}^{n-1} k^2$$

Thus, the proof "from below" follows from the equality:

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} k^2}{n^3} = \frac{1}{3} \qquad (3)$$

which is also true, since the sum in the numerator of (3) is:

$$P_{n-1} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} - n^2$$

which is still a third-degree polynomial in n with leading coefficient equal to 1/3.

Notes

1 - The model chosen for our proof provides an opportunity to show in a different way as stated by Archimedes in the proposition 15. Our proof can refer (without changing anything in the text) to a figure more general obtained by shifting arbitrarily the segment BC on its straight line, like this:

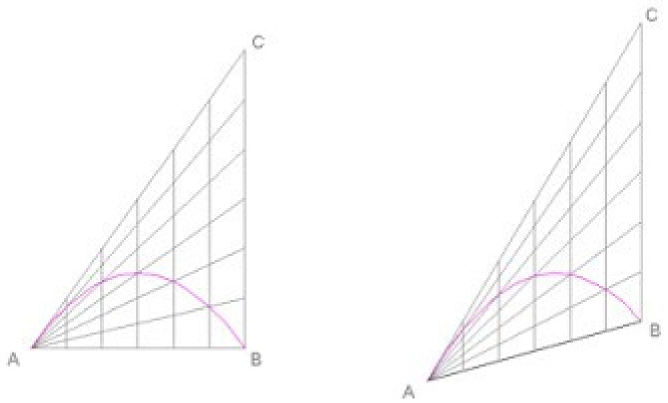


Fig. 6

In fact, the transformation does not affect any of equivalence relations between trapezoids and triangles used in the proof, which are the essence of the proof itself.

2 - The (2) can also be tested in a visual manner, in the following way: construct a right pyramid with a square base placing n^2 unit cubes at the base and going up concentrically with $(n-1)^2$, $(n-2)^2$, ... , $9, 4, 1$ cube in the top.

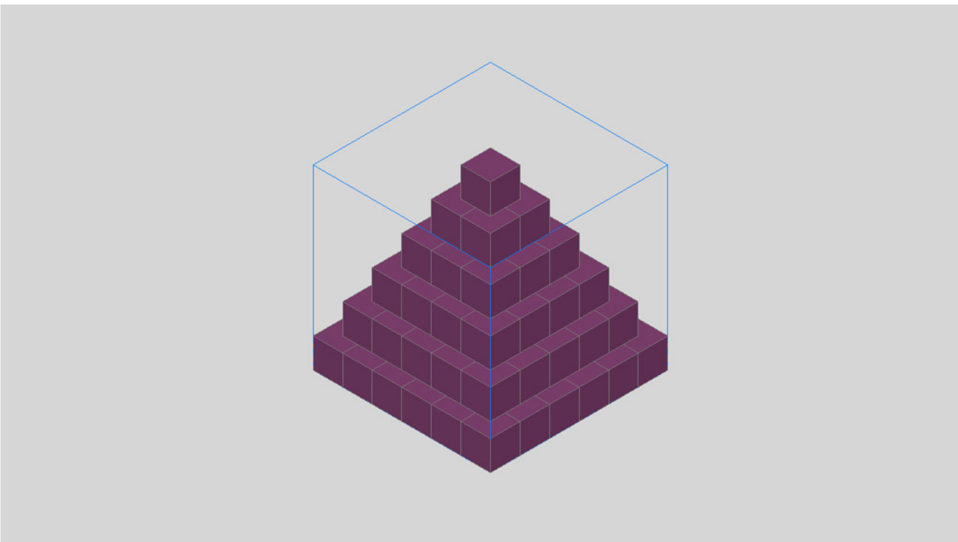


Fig. 4

This is (for $n = 6$) a geometrical representation of the *square pyramidal number* P_n, whose volume (number of unit cubes) will be given by the numerator in (2). The denominator of (2) is the volume of the cube that contains it.

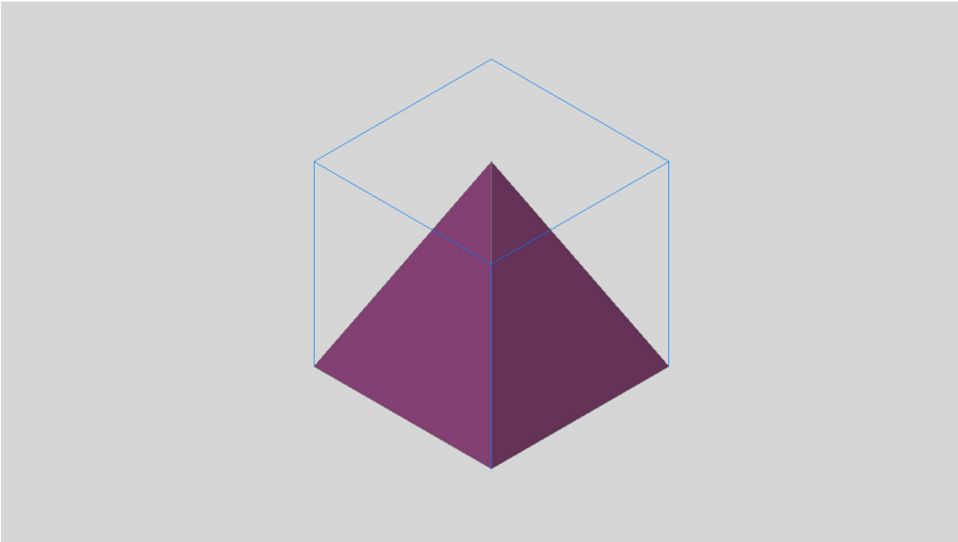


Fig. 5

At the limit of n = *infinity* the pyramid becomes "ideal" and it is known that its volume will be equal to one third of the volume of the enclosing cube.

There are therefore two different patterns described by the same relation (2). This suggests that they can be others. You can for example investigate other two figures, the cone and the circumscribed cylinder, the volumes of which are in the ratio 1 to 3.

Construct then a new solid model placing in top an unit cylinder, having equals base diameter and height, and come down concentrically with cylinders of same height, having diameters of 2, 3, 4, ... n times the diameter of the top cylinder. You can see immediately that the volumes of these cylinders grow up as the squares of natural numbers.

Continuing in the same way, with other plane figures, rather than the square and the circle, it turns out that there exist infinitely many models that describe the (2). Here the plane figures to be used are *regular polygons*, all inscribable in a circle. You will always have that the areas of these figures, and then the volumes of the layers, are variables as the squares of n.

Finally, note that the concentricity of the models is not an essential condition, that is, the same thing also applies to any model oblique (or constructed on any curved axis).

3 - What was said in algebraic terms in the box between the green dashes p. 9, can be "seen", with reference to the geometric models represented in Figures 1 and 4-5, in the following way:

A - The third degree term of the polynomial in the numerator of (2) corresponds: at the parabolic segment in Fig. 1, and at the regular pyramid (inscribed) in Fig. 5.

B - The lower degree terms of the same polynomial, are: the saw-tooths in Fig 1, and the "steps" of the pyramid in Fig. 4 (see figures in page 15).

In the limit process expressed by (2), the quantity A does not vary, while the quantity B become more and more evanescent, until it disappears completely at infinity.

4 - A further investigation of (2) leads to the following generalization:

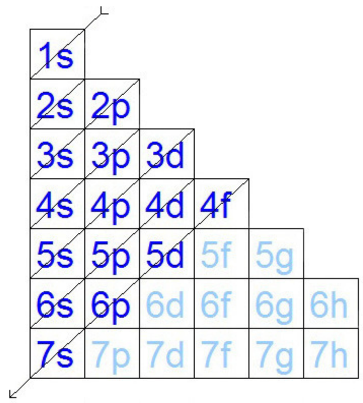
$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^m}{n^{m+1}} = \frac{1}{m+1} \qquad \forall m \in \mathbb{N} \tag{4}$$

Where, as *m* varies, the quantities in the numerator are, neatly: the *triangular number* T_n, the *square pyramidal number* P_n, the *squared triangular number* T_n^2 and other sums which you calculate using the *Faulhaber's formula*.

The limit (4) follows from the fact that a Faulhaber's polynomial of degree $m+1$ has leading coefficient equal to $1 / m + 1$. The (4) generates, as *m* varies, the convergent sequence of inverses of natural numbers: *1, 1/2, 1/3, 1/4, ..., 1 / m +1, ...*

You might at this point to rewrite all the talk of previous note 3 in terms of Faulhaber's polynomials of degree $m+1$ and related ($m+1$)-dimensional models, distinguishing in them, in the limit process (4), the invariant components by the "roughness" (corresponding to the teeth and the steps seen in note 2), which instead tends to zero at infinity.

5 - Another example of how it is possible to order data walking the grids, I found it years ago studying "Structure of matter". The figure shows a diagonal path that reproduces the order of filling, increasing energy, the quantum states of an atom:



ie the famous sequence: 1s 2s 2p 3s 3p 4s 3d 4p 5s 4d 5p 6s 4f 5d 6p 7s ….

Links

1 - See *animation* of the proof at: http://youtu.be/Lt0GI8FFQXg
2 - https://en.wikipedia.org/wiki/The_Quadrature_of_the_Parabola
3 - https://en.wikipedia.org/wiki/Faulhaber's_formula

Research Projects

1 – Search for other propositions on relationships between plane figures, of the type considered here, which can be proved in a manner similar to that used here.

2 - The quadrature of the parabola with P_n, together with the visual inspection made in note 2, are two examples of how you can use, for "practical" purposes , an entity purely "theoretical" as P_n.
See if there are (always with P_n) other cases.

Sum of the squares of the first “n” natural numbers

Abstract: Will perform here the derivation of the formula for calculating the *square pyramidal number*, which is the sum of the squares of the first n natural numbers, using a three-dimensional geometric model.

Keywords: pyramidal number, number theory, geometry, principle of mathematical induction.

Introduction

In mathematics, the square pyramidal number is a number that figuratively represents the number of spheres stacked in a pyramid with a square base. The nth number of this type is then the sum of the squares of the first n natural numbers. The formula that calculates this sum, reported in the proposition, was obtained by an algebraic process, in an indirect manner. George Polya, in his book *The mathematical discovery*, presents this solution saying it "rained from the sky", obtained algebraically by a trick, like a rabbit drawn out from the hat. In the geometric proof that follows the same result is obtained in a direct manner.

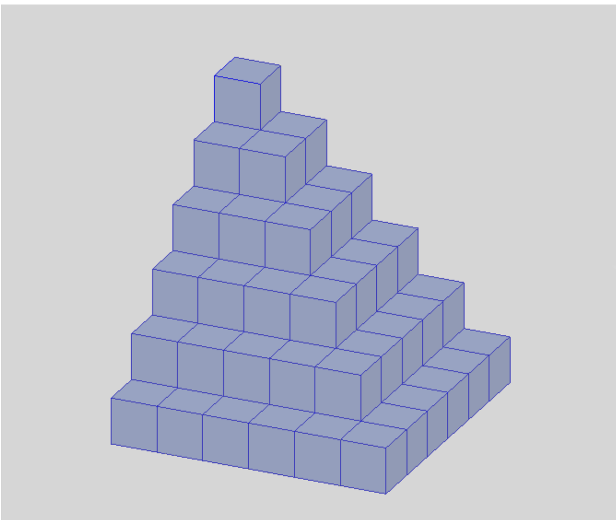
Proposition

The sum of the squares of the first n natural numbers is given by the square pyramidal number, expressed by the following formula:

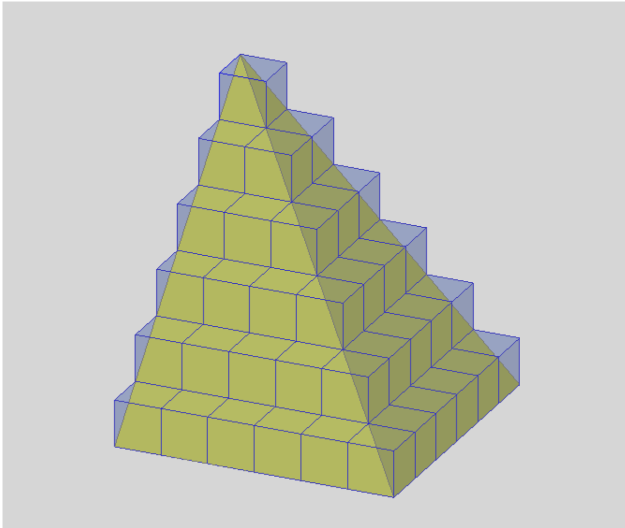
$$P_n = \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

Proof

We build a three-dimensional geometric model that represents the sum of the squares of the first 6 natural numbers P_6, using cubic bricks of unit volume (see below the linked second animation):



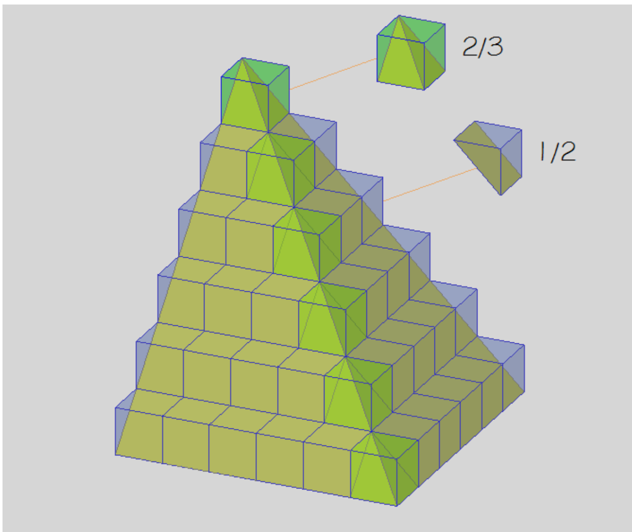
We insert now in the building a pyramid (yellow), inscribed as follows:



Let V_6 the volume of the inscribed pyramid. To obtain the total volume of the building P_6 , just add to the volume V_6 of the yellow pyramid, the excess volume that is outside of the pyramid itself.

This excess is:

- 2/3 for each unit cube placed on the central edge of the pyramid;
- 1/2 for each unit cube forming the steps of the building.



Calculating one has:

$$P_6 = V_6 + \frac{2}{3}\cdot 6 + \frac{1}{2}\cdot(2+4+6+8+10) =$$

$$= V_6 + \frac{2}{3}\cdot 6 + (1+2+3+4+5)$$

Applying the *induction principle*, we can write that, in general:

$$P_n = V_n + \frac{2n}{3} + \sum_{k=1}^{n-1} k = \frac{n^3}{3} + \frac{2n}{3} + \frac{n^2+n}{2} - n$$

that is:

$$P_n = \sum_{k=1}^{n} k^2 = \frac{2n^3+3n^2+n}{6}$$

What is the formula that we were looking.

Links

1 - The animation below shows the basic idea that gave rise to our search: https://en.wikipedia.org/wiki/File:Squares0.gif

2 - The method used in the proof applies equally well to cases of sums of squares of the first n odd and even numbers; see the animation at: http://youtu.be/BstNdX9lyqQ

3 - http://en.wikipedia.org/wiki/Square_pyramidal_number

References

George Polya (1981), *Mathematical Discovery - Vol. II*, Paperback

Sum of squares of the first "*n*" odd and even numbers

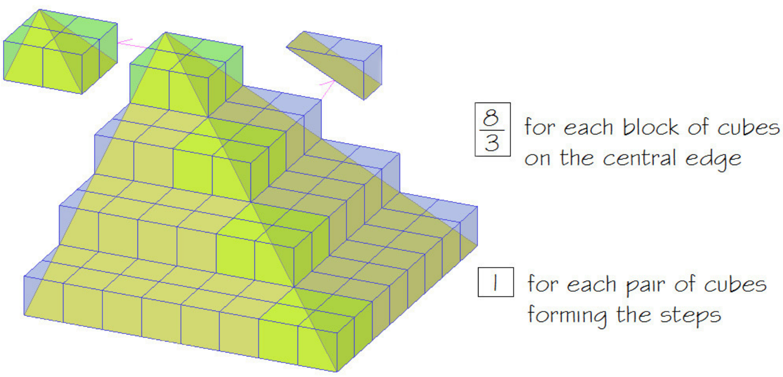
The method used in the previous proof, that we will say "*method of the inscribed pyramid*", applies equally well to the case of sums of squares of the first *n* odd and even numbers.

Sum of squares of the first "*n*" even numbers

The sum of the squares of the first *n* even numbers is obtained from the well-known formula:

$$\sum_{k=1}^{n}(2k)^2 = \frac{2(2n^3+3n^2+n)}{3}$$

Proceeding with the introduced method you get to the end, for exceeding volumes, the following situation:



The volume of building S4 is calculated by adding to the volume V4 of the inscribed pyramid, the total volume of excess parts:

$$S_4 = V_4 + \frac{8}{3}\cdot 4 + 2\cdot(2+4+6)$$

We can then write that in general:

$$S_n = V_n + \frac{8n}{3} + 2(n^2-n) = \frac{(2n)^3}{6} + \frac{8n}{3} + 2(n^2-n)$$

that is:

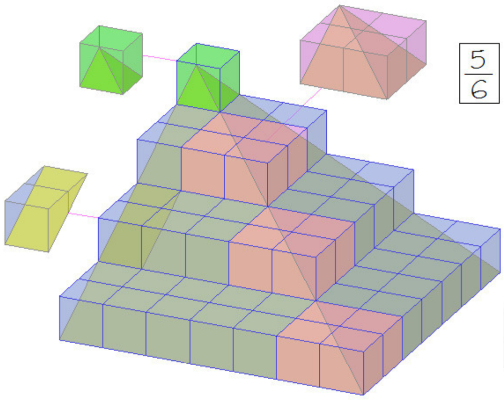
$$\sum_{k=1}^{n}(2k)^2 = \frac{2(2n^3+3n^2+n)}{3}$$ which is the formula that we were looking.

Sum of squares of the first "*n*" odd numbers

The sum of the squares of the first *n* odd numbers is obtained from the formula:

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$$

Even here, proceeding with the same method, are obtained at the end the following excess volumes:



The volume of building S'4 is calculated by adding to the volume V'4 of the inscribed pyramid, the total volume of excess parts:

$$S'_4 = V'_4 + \frac{8}{3}\cdot 3 + \frac{5}{6} + 2\cdot(1+3+5) =$$

We can then write that in general:

$$S'_n = V'_n + \frac{8(n-1)}{3} + \frac{5}{6} + 2(n-1)^2 = \frac{(2n-1)^3}{6} + \frac{8(n-1)}{3} + \frac{5}{6} + 2(n-1)^2$$

that is:

$$\sum_{k=1}^{n}(2k-1)^2 = \frac{4n^3-n}{3}$$ which is the formula that we were looking.

See animation: http://youtu.be/BstNdX9lyqQ

Sum of cubes of the first "n" natural numbers

Abstract: We will demonstrate here the Nicomachus's theorem on the sum of the cubes of the first n natural numbers, using the manipulation of a three-dimensional geometric model.

Keywords: Nicomachus, triangular number, number theory, arithmetic, geometry, principle of mathematical induction.

Introduction

In the number theory, the sum of the first n cubes is given by the square of the nth triangular number, that is,

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2$$

This identity is sometimes called *Nicomachus's theorem*, named after the Greek mathematician of the Hellenistic age, Nicomachus of Gerasa, which proved it arithmetically. Many mathematicians have studied this equality, demonstrating it in many different ways. The idea of visually demonstrate the Nicomachus's identity is not new. Roger B. Nelsen, in his work *Proofs without Words* (1993) presents seven different versions. The advantage of visual demonstrations is to provide sometimes, as in the present work, a graphic evidence of the solution.

Proposition

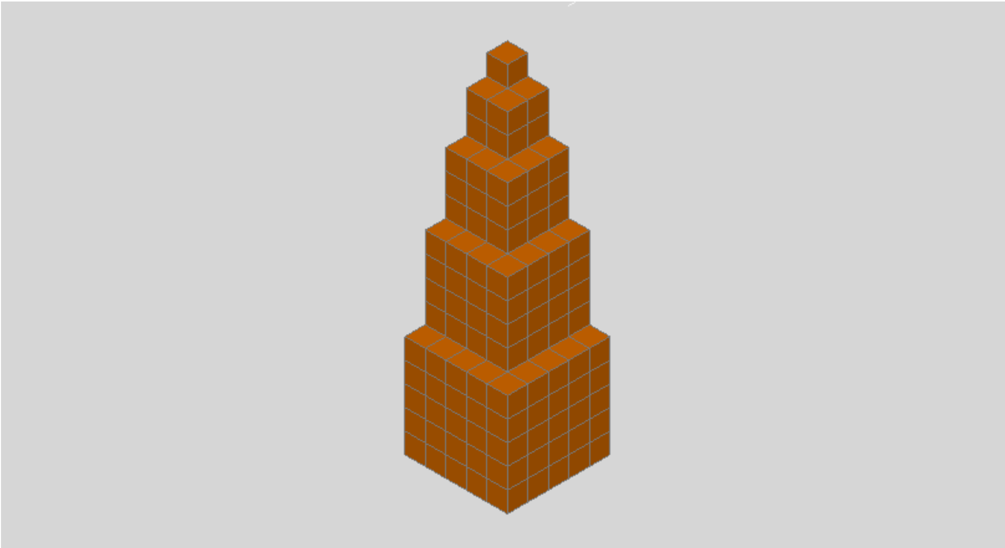
The sum of the cubes of the first n natural numbers is given by the square of the nth triangular number:

$$\sum_{k=1}^{n} k^3 = T_n^2 = \left(\sum_{k=1}^{n} k \right)^2$$

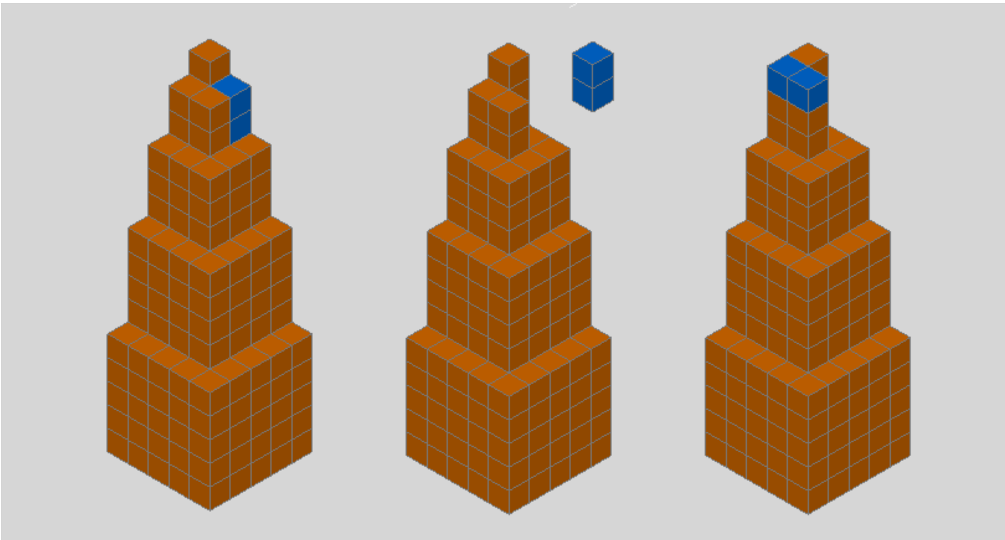
where the triangular number is equal to the sum of the first n natural numbers.

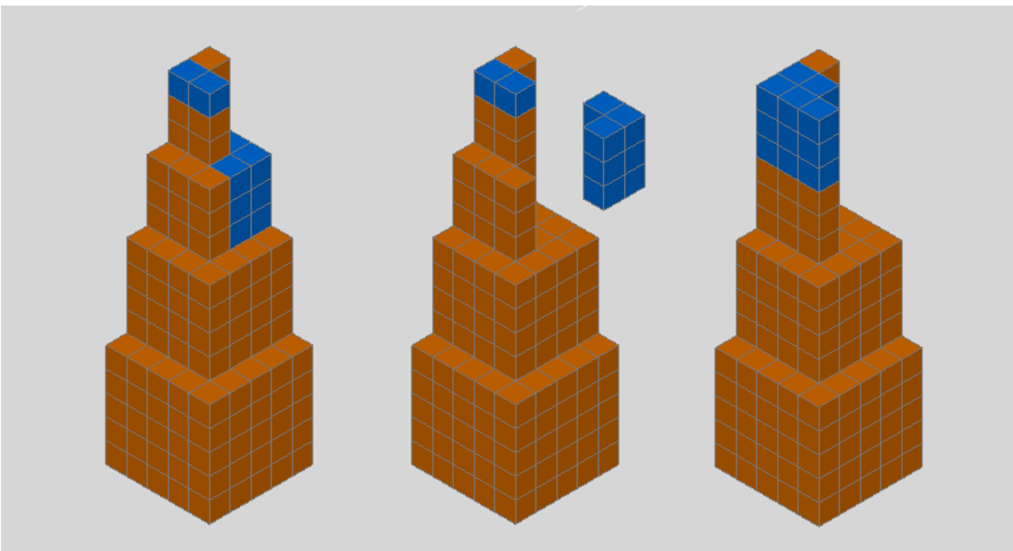
Proof

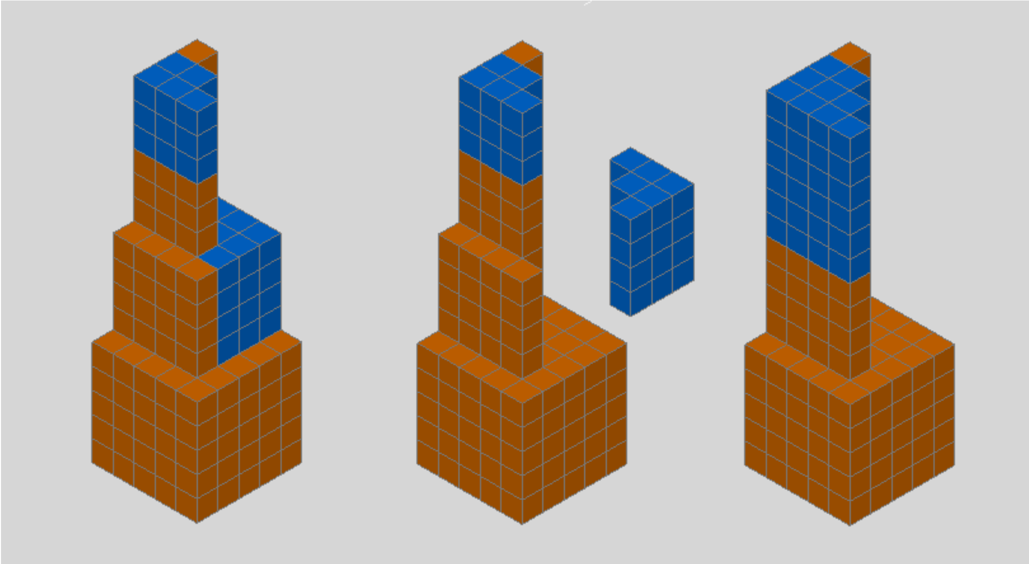
We build a three-dimensional geometric model that represents the sum of the cubes of the first 5 natural numbers, using cubic bricks of unit volume, in the following way (see below the linked second animation):

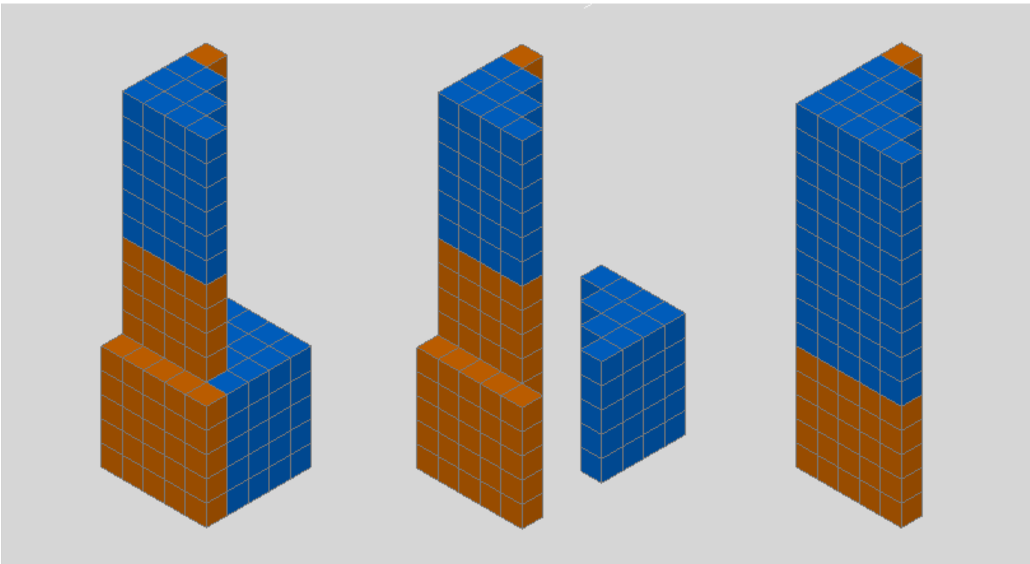


In an attempt to obtain a figure equivalent to this model, which gives evidence of the identity to prove, we operate on the model an inductive transformation, moving the unit cubes as follows:



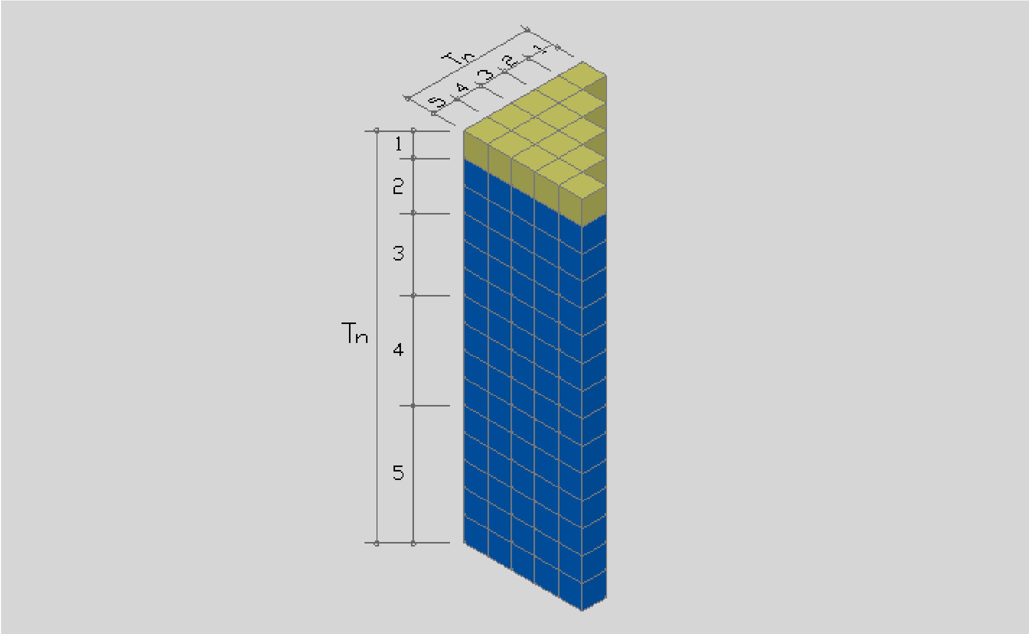






The inductivity of the process lies in the fact that, in each cube of the sum, the unit cubes to move are neatly arranged in $1 + 2 + 3 + \ldots + (k-1)$ columns, each of height k.

The final result of the transformation has been always, for any n, a pseudo-parallelepiped whose base is a geometrical representation of the triangular number T_n, and whose height is the number T_n itself, which remains unchanged during the transformation.



It is thus evident that, the total number of unit cubes, which gives the sum of the cubes of the first n natural numbers, is given by:

$$\sum_{k=1}^{n} k^3 = T_n{}^2 = \left(\sum_{k=1}^{n} k \right)^2$$

that is, the identity that one wanted to prove.

Links

1 - The animation below shows the basic idea that gave rise to our search
https://en.wikipedia.org/wiki/File:Idea01.gif

2 - The method used in the proof applies equally well to cases of sums of the cubes of the first n odd and even numbers; see the animation at:
http://youtu.be/HGCBhnuuHvM

3 - http://en.wikipedia.org/wiki/Squared_triangular_number

References

Nelsen, Roger B. (1993), *Proofs without Words*, Cambridge University

Sum of the cubes of the first " *n* " odd and even numbers

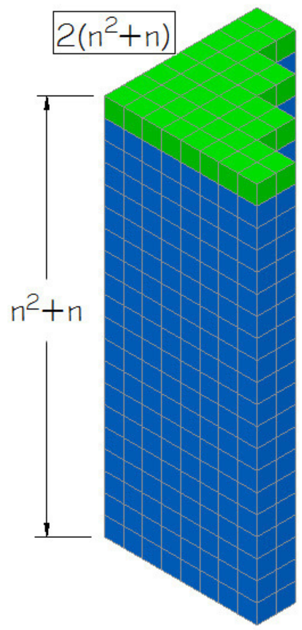
The method used in the previous proof, that we will say "*method of successive transformations*", applies equally well to cases of sums of cubes of the first *n* odd and even numbers.

Sum of the cubes of the first "n" even numbers

The sum of the cubes of the first *n* even numbers is obtained from the known formula:

$$\sum_{k=1}^{n}(2k)^3 = 2(n^2+n)^2$$

Proceeding with the introduced method is obtained, at the end of the transformation, the following figure:



that is, a pseudo-parallelepiped having its base formed by

$$4\cdot(1+2+3+....+n) = 2(n^2+n)$$

unit cubes, and height (which remains unchanged) amounting to

$$(n^2+n)$$

unit cubes.

Therefore, the volume of the figure, ie the sum they were looking, is:

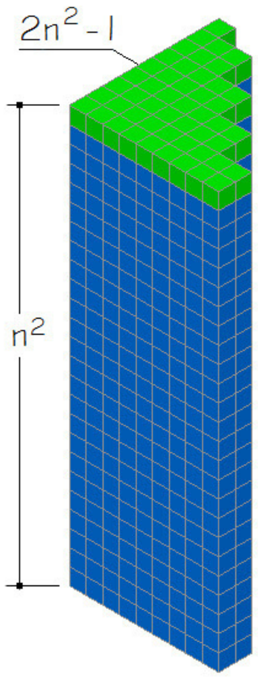
$$\sum_{k=1}^{n}(2k)^3 = 2(n^2+n)^2$$

Sum of the cubes of the first "n" odd numbers

The sum of the cubes of the first *n* odd numbers is obtained from the formula:

$$\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$$

Even here, proceeding with the transformation, is obtained at the end the figure:



From which, by calculating the number of unit cubes, is:

$$\sum_{k=1}^{n}(2k-1)^3 = n^2(2n^2-1)$$ which is the formula they were looking.

See animation at link: http://youtu.be/HGCBhnuuHvM

Sum of the first "n" sums of powers

We consider the sums of powers of successive integers:

$$\sum_{k=1}^{n} k^m = 1^m + 2^m + ... + n^m$$

which, as we know, are calculated with the *Faulhaber*'s formulas, as follows:

$$\sum_{k=1}^{n} k = \tfrac{1}{2}\left(n^2 + n\right)$$

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}\left(2\,n^3 + 3\,n^2 + n\right)$$

$$\sum_{k=1}^{n} k^3 = \tfrac{1}{4}\left(n^4 + 2\,n^3 + n^2\right)$$

$$\sum_{k=1}^{n} k^4 = \tfrac{1}{30}\left(6\,n^5 + 15\,n^4 + 10\,n^3 - n\right)$$

$$\sum_{k=1}^{n} k^5 = \tfrac{1}{12}\left(2\,n^6 + 6\,n^5 + 5\,n^4 - n^2\right)$$

.... (the table continues indefinitely).

Each of these formulas generates, as n varies, an increasing numerical sequence, of the type of that obtained for $m = 2$:

1, 5, 14, 30, 55, 91, 140, 204, 285, ……

that is the succession of the *square pyramidal numbers*.

We aim to find a way to calculate, given any of these sequences, the sum of its first n terms, that is, the sum:

$$\sum_{k=1}^{n} (1^m + 2^m + ... + n^m) \tag{1}$$

An opportunity to obtain this is offered by the following table:

1^m	1^m	1^m	1^m	1^m	1^m	1^m	1^m	...	1^m	1^m
2^m	2^m	2^m	2^m	2^m	2^m	2^m	2^m	...	2^m	2^m
3^m	3^m	3^m	3^m	3^m	3^m	3^m	3^m	...	3^m	3^m
4^m	4^m	4^m	4^m	4^m	4^m	4^m	4^m	...	4^m	4^m
5^m	5^m	5^m	5^m	5^m	5^m	5^m	5^m	...	5^m	5^m
6^m	6^m	6^m	6^m	6^m	6^m	6^m	6^m	...	6^m	6^m
7^m	7^m	7^m	7^m	7^m	7^m	7^m	7^m	...	7^m	7^m
:	:	:	:	:	:	:	:	...	:	:
$(n-1)^m$	$(n-1)^m$	$(n-1)^m$	$(n-1)^m$	$(n-1)^m$	$(n-1)^m$	$(n-1)^m$	$(n-1)^m$	...	$(n-1)^m$	$(n-1)^m$
n^m	n^m	n^m	n^m	n^m	n^m	n^m	n^m	...	n^m	n^m

We describe the contents of the table:

- By summing the content of each column (black + red boxes), we obtain the sum of the first n m-th powers, that (in tribute to Faulhaber) we denote by F_m :

$$F_m = \sum_{k=1}^{n} k^m$$

and the contents of the entire table will be then:

$$(n+1)F_m = (n+1)\sum_{k=1}^{n} k^m$$

- The black section contains, in each row, the amount:

$$k^m \cdot k = k^{(m+1)}$$

By summing the contents of all rows one obtains the sum of the first n $(m+1)$-th powers:

$$F_{(m+1)} = \sum_{k=1}^{n} k^{(m+1)}$$

- The red section contains, in the columns, the sequence of F_m sums. By summing the contents of all columns one obtains the quantity that we want to calculate, namely (1):

$$\sum_{k=1}^{n}(1^m + 2^m + ...+ n^m) = \sum_{k=1}^{n} F_m$$

The quantity that we seek is then obtained by subtracting to the content of the entire table, the content of the black boxes, that is:

$$\sum_{k=1}^{n} F_m = (n+1)F_m - F_{(m+1)} \tag{2}$$

By performing algebraic calculations for m = 1, 2, 3, you get:

***m* = 1**

$$\sum_{k=1}^{n} F_1 = (n+1)F_1 - F_2$$

$$= (n+1)\frac{n^2+n}{2} - \frac{2n^3+3n^2+n}{6} = \boxed{\frac{n^3+3n^2+2n}{6}}$$

***m* = 2**

$$\sum_{k=1}^{n} F_2 = (n+1)F_2 - F_3$$

$$= (n+1)\frac{2n^3+3n^2+n}{6} - \left[\frac{n^2+n}{2}\right]^2 = \boxed{\frac{n^4+4n^3+5n^2+2n}{12}}$$

***m* = 3**

$$\sum_{k=1}^{n} F_3 = (n+1)F_3 - F_4$$

$$= (n+1)\left[\frac{n^2+n}{2}\right]^2 - \frac{6n^5+15n^4+10n^3-n}{30} = \boxed{\frac{3n^5+15n^4+25n^3+15n^2+2n}{60}}$$

Tests were positive with Excel.

Below is the scan test performed for m = 3, up to n = 7:

n	n^3	F_3	A	B
1	1	1	1	1
2	8	9	10	10
3	27	36	46	46
4	64	100	146	146
5	125	225	371	371
6	216	441	812	812
7	343	784	1596	1596

Column A contains the amounts calculated as follows: box to the left + above box.
Column B contains the amounts calculated by entering the formula to verify.

Polynomial expressions generated by (2) are the natural extension of those listed at the beginning. The general formula for obtaining them in a direct way is written, in the compact notation of Faulhaber's formula, in the following way:

$$\sum_{k=1}^{n} F_m = (n+1)F_m - F_{(m+1)}$$

$$= \frac{n+1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(n+1)^{m+1-k} - \frac{1}{m+2}\sum_{k=0}^{m+1}\binom{m+2}{k}B_k(n+1)^{m+2-k} \qquad (3)$$

where the B_k quantities are the *Bernoulli numbers*.

This formula calculates, for each natural number n and for each power m, the sum of the first n sums of powers.

Animation at: https://www.youtube.com/watch?v=PnIEPqFtcQc

A relationship between figurate numbers

The "square pyramidal number" can be decomposed into the sum of two tetrahedral numbers less a triangular number, ie, in the following way:

$$P_n = 2\Theta_n - T_n \tag{1}$$

You can see it easily in two different ways:

- You can put in a column two sequences of tetrahedral numbers:
  1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, ……
  1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, ……
  and a sequence of triangular numbers:
  1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, ……
  and subtract the latter from the sum of the first two, obtaining:
  1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, ……

- Or, replacing the second member of (1) with the solving formulas:

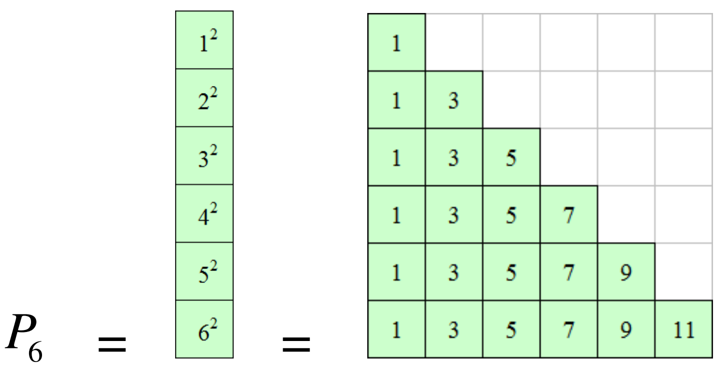
$$P_n = 2\frac{n(n+1)(n+2)}{6} - \frac{n(n+1)}{2} = \frac{2n^3+3n^2+n}{6} \tag{2}$$

But doing so we perform simple checks.

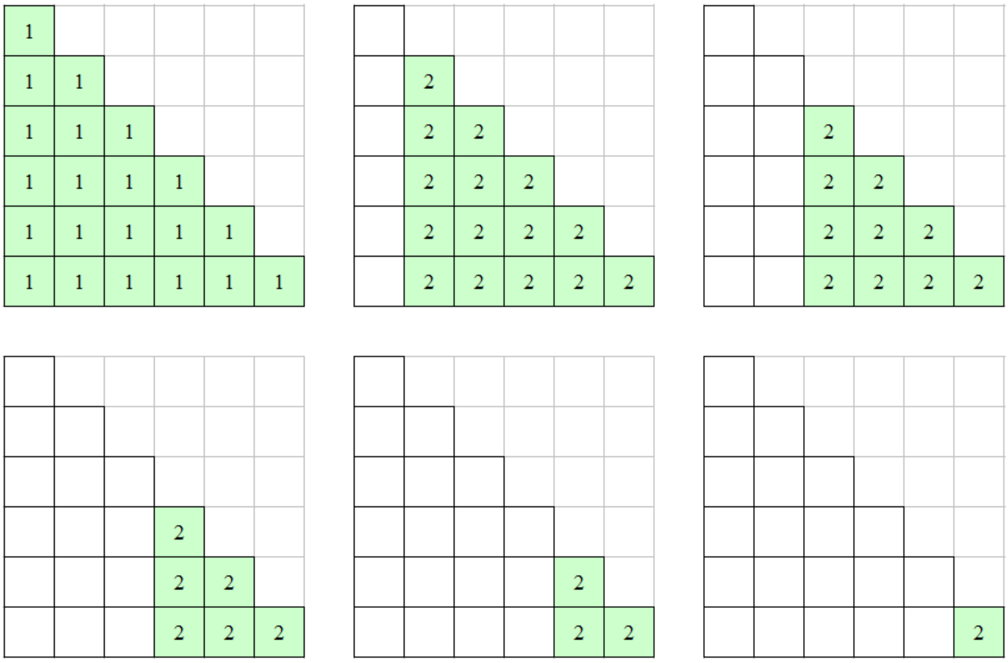
I believe that there are many relations of the type proposed in (1). They may search with a calculation program that analyzes correspondences between the values of sequences of figured numbers, as we did in the first test. However, the possible results of such research should be explained and proved for each n.

I show you how I found out the proposition, giving at the same time the proof of it.

Consider the scheme used to p. 8 to represent the square pyramidal number P_6:



This scheme can be seen as the superposition of six homogeneous "layers" :



which represent, in terms of triangular numbers, the following quantities:

T_6, $2T_5$, $2T_4$, $2T_3$, $2T_2$, $2T_1$

therefore, we can write :

$$P_6 = T_6 + 2(T_5 + T_4 + T_3 + T_2 + T_1) = T_6 + 2\sum_{k=1}^{5} T_k$$

But the sum of the first five triangular numbers is the tetrahedral number Θ_5, then:

$$P_6 = T_6 + 2\Theta_5 = T_6 + 2(\Theta_6 - T_6) = 2\Theta_6 - T_6$$

Even here, the generalization follows from the fact that the passage, by a number n to the next, is an *inductive* process which is realized by adding:

- a line with the sequence of $n+1$ odd numbers, that is P_{n+1}, in the scheme of P_n of the first figure;
- two layers, T_n e T_{n+1} (which as you know are equivalent to the P_{n+1} added at the previous point), in the second figure.

One can therefore say that, in general, the square pyramidal number P_n can be expressed as:

$$P_n = 2\Theta_n - T_n$$

What is the relation (1) that we proposed.

Is famous the history of algebraic derivation (see page 15) of the formula for P_n, the first member of (2).
I was wondering if, having made the chronological checks on the output of the three component formulas, (2) itself (along with the demo of this article) can not be regarded as "another way" to get the formula for the sum P_n.

The Goldbach's conjecture with Excel

In number theory, the "Goldbach's conjecture" is an old unsolved problem which states:

"each even number greater than 2 can be written as the sum of two prime numbers, not necessarily distinct".

To get an idea about the validity of the conjecture, I conducted an exploratory survey, using the usual tools at my disposal.

We define "Goldbach partitions" all the different ways to write an even number N as the sum of two prime numbers:

$$\mathbf{N = p + q}$$

Then, Goldbach's conjecture states that every even number N has at least one Goldbach partition.

Using Excel, I constructed a kind of "addition" table, to obtain all the possible sums between pairs of prime numbers, in the following way:

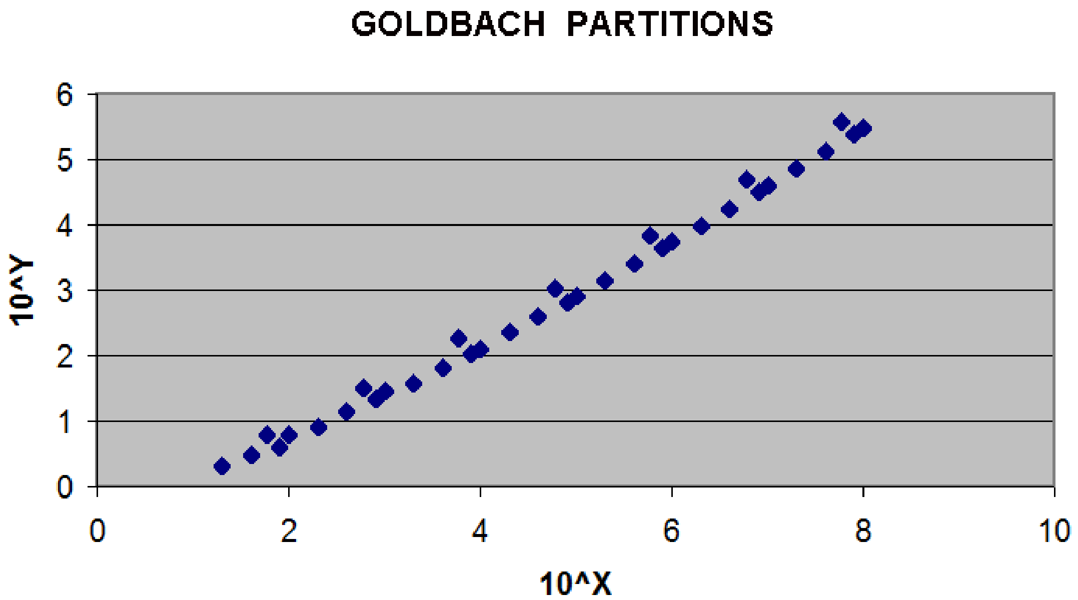
	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97
2	4																								
3		6	8	10	14	16	20	22	26	32	34	40	44	46	50	56	62	64	70	74	76	82	86	92	100
5			10	12	16	18	22	24	28	34	36	42	46	48	52	58	64	66	72	76	78	84	88	94	
7				14	18	20	24	26	30	36	38	44	48	50	54	60	66	68	74	78	80	86	90	96	
11					22	24	28	30	34	40	42	48	52	54	58	64	70	72	78	82	84	90	94	100	
13						26	30	32	36	42	44	50	54	56	60	66	72	74	80	84	86	92	96		
17							34	36	40	46	48	54	58	60	64	70	76	78	84	88	90	96	100		
19								38	42	48	50	56	60	62	66	72	78	80	86	90	92	98			
23									46	52	54	60	64	66	70	76	82	84	90	94	96				
29										58	60	66	70	72	76	82	88	90	96	100					
31											62	68	72	74	78	84	90	92	98						
37												74	78	80	84	90	96	98							
41													82	84	88	94	100								
43														86	90	96									
47															94	100									
53																									
59																									
61																									
67																									
71																									
73																									
79																									
83																									
89																									
97																									

4	1	52	3	1000	28
6	1	54	5		
8	1	56	3	10000	127
10	2	58	4		
12	1	60	6	100000	810
14	2	62	3		
16	2	64	5	1000000	5402
18	2	66	6		
20	2	68	2	10000000	38807
22	3	70	5		
24	3	72	6	100000000	291400
26	3	74	5		
28	2	76	5		
30	3	78	7		
32	2	80	4		
34	4	82	5		
36	4	84	8		
38	2	86	5		
40	3	88	4		
42	4	90	9		
44	3	92	4		
46	4	94	5		
48	5	96	7		
50	4	98	3		
		100	6		

As you can see, the table provides an even number N at each intersection between any two odd numbers, and can be extended indefinitely. The colors facilitates counting of Goldbach partitions for even numbers N from 4 to 100. The results are reported in the first two green columns to the right.

Researching on the Internet, I discovered at the site of Prof. Bo, a very useful Javascript program to find the Goldbach partitions of individual numbers. I have therefore added, in the previous figure, the number of partitions for N = 10^3, 10^4, 10^5, 10^6, 10^7 and 10^8.

With the same program I then calculated the partitions for some intermediate values to previous, resulting in a log-log plot the following XY scatter:



The graph shows a distribution of the Goldbach partitions around a curve that has a slight upward concavity, index of continuous growth. Therefore, partitions appear to be many, at each even number N which is considered, and it seems easy and safe the construction of at least a complete sequence of even numbers up to the number N regarded.

Using the addition table, I have personally tested (spot) a sequential count, which runs within a very thin range of ordinates (few hundred units) until the abscissa 10^{12}! I show you an excerpt of it around 10^5. I took the prime numbers from the Javascript program inserted in the links.

	99089	99103	99109	99119	99131	99133	99137	99139	99149	99173	99181	99191	99223	99233	99241	99251	99257	99259	99277	99289	99317	99347	99349	99367
3	99092	99106	99112	99122	99134	99136	99140	99142	99152	99176	99184	99194	99226	99236	99244	99254	99260	99262	99280	99292	99320	99350	99352	99370
5	99094	99108	99114	99124	99136	99138	99142	99144	99154	99178	99186	99196	99228	99238	99246	99256	99262	99264	99282	99294	99322	99352	99354	99372
7	99096	99110	99116	99126	99138	99140	99144	99146	99156	99180	99188	99198	99230	99240	99248	99258	99264	99266	99284	99296	99324	99354	99356	99374
11	99100	99114	99120	99130	99142	99144	99148	99150	99160	99184	99192	99202	99234	99244	99252	99262	99268	99270	99288	99300	99328	99358	99360	99378
13	99102	99116	99122	99132	99144	99146	99150	99152	99162	99186	99194	99204	99236	99246	99254	99264	99270	99272	99290	99302	99330	99360	99362	99380
17	99106	99120	99126	99136	99148	99150	99154	99156	99166	99190	99198	99208	99240	99250	99258	99268	99274	99276	99294	99306	99334	99364	99366	99384
19	99108	99122	99128	99138	99150	99152	99156	99158	99168	99192	99200	99210	99242	99252	99260	99270	99276	99278	99296	99308	99336	99366	99368	99386
23	99112	99126	99132	99142	99154	99156	99160	99162	99172	99196	99204	99214	99246	99256	99264	99274	99280	99282	99300	99312	99340	99370	99372	99390
29	99118	99132	99138	99148	99160	99162	99166	99168	99178	99202	99210	99220	99252	99262	99270	99280	99286	99288	99306	99318	99346	99376	99378	99396
31	99120	99134	99140	99150	99162	99164	99168	99170	99180	99204	99212	99222	99254	99264	99272	99282	99288	99290	99308	99320	99348	99378	99380	99398
37	99126	99140	99146	99156	99168	99170	99174	99176	99186	99210	99218	99228	99260	99270	99278	99288	99294	99296	99314	99326	99354	99384	99386	99404
41	99130	99144	99150	99160	99172	99174	99178	99180	99190	99214	99222	99232	99264	99274	99282	99292	99298	99300	99318	99330	99358	99388	99390	99408
43	99132	99146	99152	99162	99174	99176	99180	99182	99192	99216	99224	99234	99266	99276	99284	99294	99300	99302	99320	99332	99360	99390	99392	99410
47	99136	99150	99156	99166	99178	99180	99184	99186	99196	99220	99228	99238	99270	99280	99288	99298	99304	99306	99324	99336	99364	99394	99396	99414
53	99142	99156	99162	99172	99184	99186	99190	99192	99202	99226	99234	99244	99276	99286	99294	99304	99310	99312	99330	99342	99370	99400	99402	99420
59	99148	99162	99168	99178	99190	99192	99196	99198	99208	99232	99240	99250	99282	99292	99300	99310	99316	99318	99336	99348	99376	99406	99408	99426
61	99150	99164	99170	99180	99192	99194	99198	99200	99210	99234	99242	99252	99284	99294	99302	99312	99318	99320	99338	99350	99378	99408	99410	99428
67	99156	99170	99176	99186	99198	99200	99204	99206	99216	99240	99248	99258	99290	99300	99308	99318	99324	99326	99344	99356	99384	99414	99416	99434
71	99160	99174	99180	99190	99202	99204	99208	99210	99220	99244	99252	99262	99294	99304	99312	99322	99328	99330	99348	99360	99388	99418	99420	99438

At each even number counted is its "minimum" partition, obtainable with the partition program seen above.
In the test tables it can be noted a surprising density of useful data around the axis of abscissas, which allows an easy sequential count of even numbers. This fact seems to be due to two different factors that appear to act concomitantly:

- the (average) distance between two consecutive primes grows very slowly with increasing N, being of the same order of size of log N;
- as N grows you meet, with considerable frequency, twin primes (in yellow).

For effect of such combined action, it happens that the abscissas thins out and accumulate continuously, so that the data are distributed within an area bounded superiorly by the abscissa axis and inferiorly by a curve with a sinusoidal shape, in turn enveloped, approximately, by the above function log N. You might consider such a setup in search of series statistical and probabilistic arguments in support of the Goldbach conjecture.

The most mathematicians believed that the conjecture is true. If it were not, there would at least an even number that is not the sum of two primes, that is a *counter-example*. To find it, you could use the partition program (appropriately limited to the calculation of minimum partitions, or the top of each list) to see if, at even numbers ever larger, there is at least one Goldbach partition. The research carried so far has not provided any counter-example: to date has been verified that every even number less than 4×10^{18} is the sum of two prime numbers.

However, approaches to the problem of Goldbach like the one we used can only suggest an opinion (in favor of the truth in our case) but do not prove the conjecture rigorously, ie for each N.

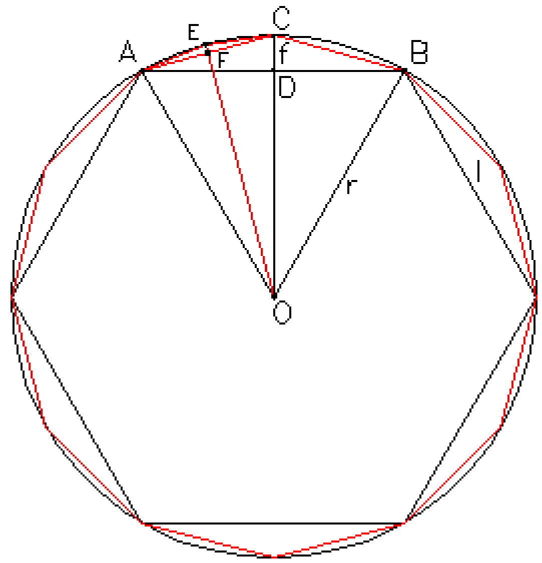
The problem is therefore still open.

Links

- http://utenti.quipo.it/base5/numeri/jsgolbachpartiz.htm
- http://www.walter-fendt.de/m14i/primes_i.htm

Measurement of the circle

We propose here to set up and run, on a spreadsheet, the calculation of π, employing the idea of Archimedes of inscribing in a circle, at first the regular polygon with 6 sides (regular hexagon), then (by halving the center angles) that of 12 sides, then that of 24, 48 and 96 sides, calculating for this latter a perimeter equal:
"... to three times the diameter and a certain portion of it that is smaller than a seventh and larger than 10/71 times of the same diameter "
which is the approximate value suggested by Archimedes.



The circle in the figure has unit diameter and inscribe in it the regular hexagon. Halve the angle AOB by the OC, then halve the angle AOC by OE and continue so indefinitely, obtaining the sequence of regular polygons of 12, 24, 48, ... etc.. sides, inscribed in the circumference, we associate with positive integers n (to n = 1 the hexagon, to n = 2 the dodecagon, etc.).

The arrow CD of the arc AB, denoted by f, is:

$$CD = OB - (OB^2 - DB^2)^{1/2}$$ that is:

$$f_1 = r - (r^2 - (l_1 / 2)^2)^{1/2}$$

Where r is the radius of the circle and l_1 the side of the hexagon $(r = l_1 = 0,5)$.

We have, in general:

$$f_n = r - (r^2 - (l_n / 2)^2)^{1/2}$$

and the lengths of the sides of the polygons are calculated in succession:

$$l_{n+1} = (f_n{}^2 + (l_n / 2)^2)^{1/2}$$

Entering formulas in a spreadsheet, as follows:

n	N. lati	f_n	l_n	p_n
1	6	=0,5-(0,5^2-(D2/2)^2)^0,5	0,5	=+D2*B2
=+A2+1	=+B2*2	=0,5-(0,5^2-(D3/2)^2)^0,5	=+(C2^2+(D2/2)^2)^0,5	=+D3*B3
=+A3+1	=+B3*2	=0,5-(0,5^2-(D4/2)^2)^0,5	=+(C3^2+(D3/2)^2)^0,5	=+D4*B4
=+A4+1	=+B4*2	=0,5-(0,5^2-(D5/2)^2)^0,5	=+(C4^2+(D4/2)^2)^0,5	=+D5*B5
=+A5+1	=+B5*2	=0,5-(0,5^2-(D6/2)^2)^0,5	=+(C5^2+(D5/2)^2)^0,5	=+D6*B6
=+A6+1	=+B6*2	=0,5-(0,5^2-(D7/2)^2)^0,5	=+(C6^2+(D6/2)^2)^0,5	=+D7*B7
:	:	:	:	:

you get the table:

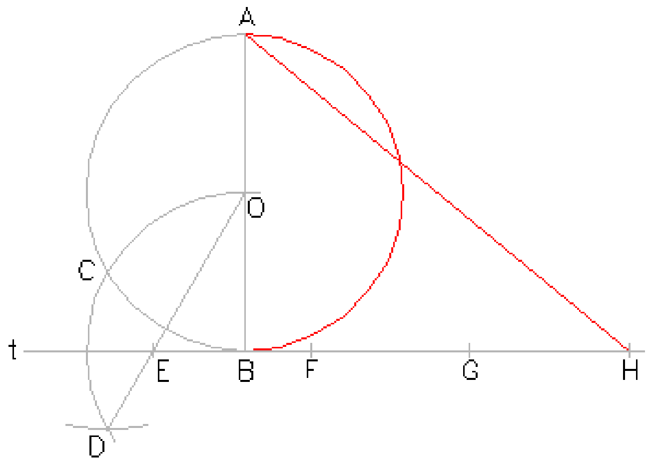
n	N. lati	f_n	l_n	p_n
1	6	0,06698729810778	0,50000000000000	3,00000000000000
2	12	0,01703708685547	0,25881904510252	3,10582854123025
3	24	0,00427756931309	0,13052619222005	3,13262861328124
4	48	0,00107053838070	0,06540312923014	3,13935020304687
5	96	0,00026770626182	0,03271908282178	3,14103195089051
6	192	0,00006693104522	0,01636173162649	3,14145247228546
7	384	0,00001673304130	0,00818113960394	3,14155760791186
8	768	0,00000418327782	0,00409060402623	3,14158389214832
9	1536	0,00000104582055	0,00204530629116	3,14159046322805
10	3072	0,00000026145521	0,00102265368034	3,14159210599927
11	6144	0,00000006536381	0,00051132690701	3,14159251669216
12	12288	0,00000001634095	0,00025566346186	3,14159261936538
13	24576	0,00000000408524	0,00012783173198	3,14159264503369
14	49152	0,00000000102131	0,00006391586612	3,14159265145077
15	98304	0,00000000025533	0,00003195793308	3,14159265305504
16	196608	0,00000000006383	0,00001597896654	3,14159265345610
17	393216	0,00000000001596	0,00000798948327	3,14159265355637
18	786432	0,00000000000399	0,00000399474164	3,14159265358144
19	1572864	0,00000000000100	0,00000199737082	3,14159265358770
20	3145728	0,00000000000025	0,00000099868541	3,14159265358927
21	6291456	0,00000000000006	0,00000049934270	3,14159265358966
22	12582912	0,00000000000002	0,00000024967135	3,14159265358976
23	25165824	0,00000000000000	0,00000012483568	3,14159265358979
:	:	:	:	:

The last column of the table contains the sequence of p_n values, the perimeter of the nth regular polygon inscribed in the circle of unit diameter.
It 'clear that, as the number of sides of the polygon tends to infinity:

$$\lim_{n \to \infty} p_n = \pi$$

Quadrature of the circle

The problem of rectification of the circumference and the equivalent of quadrature of the circle, cannot be solved in the manner of the ancient Greeks, that is, with the exclusive use of ruler and compass.
However, there are approximate solutions to this problem. One of these was discovered in 1685 by Adam Kochansky, a Jesuit who worked as librarian of King John of Poland.



The figure shows the " Kochansky's approximate quadrature of the circle". In effect it is (red lines) the rectification of a semicircle. This geometrical construction was performed using AutoCAD, in the following way:

We track:
- a circle of center O and diameter AB and a straight line tangent to the circle at the point B;
- with center B, an arc of radius BO that intersects the circle at the point C;
- with center C and with the same radius, another arc, which intersects the first at point D;
- the DO straight line, which intersects the line t at the point E.

Then, keeping fixed the compass opening on the initial value BO, were subsequently reported, starting from E, the points F, G, and H (EF = FG = GH = BO). Finally they joined the points A and H.
Having performed the construction with a compass opening BO = 1, according to the calculations of Kochansky, the segment AH thus obtained should measure: (1)

AH = 3,141533

which is the approximation found by Kochansky, with an error of about 1/16000.

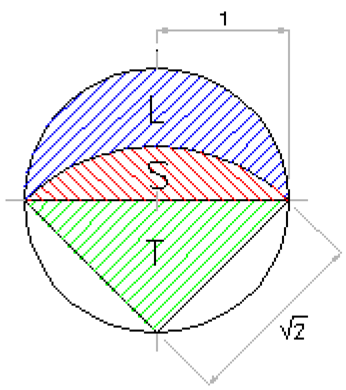
AutoCAD gives us the opportunity to immediately check this degree of accuracy. Using the "List" command on the object AH, you get it in "Text Window" the following output:

```
LINE        Layer: 0   Space: Model space
            Color: 1 (red) Linetype: BYLAYER
            Handle = 56
      from point, X= 2.422650 Y= 0.000000 Z= 0.000000
        to point, X= 0.000000 Y= 2.000000 Z= 0.000000
Length = 3.14153334, Angle in XY Plane = 140
           Delta X =-2.422650, Delta Y = 2.000000, Delta Z =
0.000000
```

The lune of Hippocrates

We have seen that the circle can be squared only in an approximate way. However, with simple geometric procedures, it is possible to square up exactly some parts of the circle, such as the "lune of Hippocrates".

Consider in the figure the "lune" (dotted blue) bounded by two circular arcs with radius 1 and $2^{1/2}$ respectively. We try to square this part of the circle, that is, determine its area through the construction (by ruler and compass) of a figure equivalent to it.



We distinguish, in addition to the above lune denoted by the letter L, the following other two figures: the circular segment S dashed red and the triangle T dashed green.

The area of the *quadrant* consisting of the figures T and S is:

$T+S = (1/4) \cdot (2^{1/2})^2 \cdot \pi = \pi/2$

The area of the semicircle consisting of the figures L and S is:

$L+S = \pi/2$

then we have: $T = \pi/2 - S$ and

$L = \pi/2 - S$

that is: $L = T$.

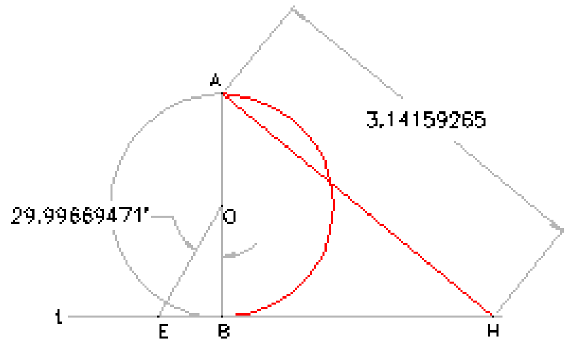
The triangle T is the figure that we wanted to build. (2)

Notes

(1) Calculating, one has:

$$EB = OB \cdot \tan 30° = 0{,}577350$$
$$BH = 3 - EB = 2{,}422650$$
$$AH = (AB^2+BH^2)^{1/2} = 3{,}141533$$

The approximation of Kochansky is therefore obtained at an angle of EOB=30°, which is constructible with ruler and compass, as required. One wonders what is the value of the EOB angle that "builds" exactly π. Still using AutoCAD and performing the construction contrary, we obtained the result shown in the following figure:



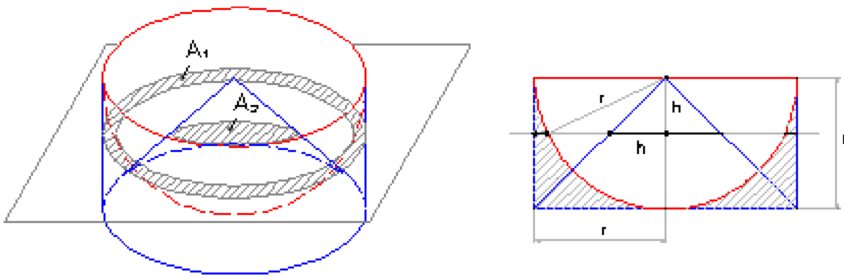
The value of EOB which gives π is very close to 30 °.

(2) If we consider that the triangle T is equivalent to a square of side = 1, the term "quadrature" is proper.

On the sphere and the cylinder

The volume of the sphere is two-thirds of the volume of a cylinder having as base a maximum circle of the sphere and for height the diameter of it.

We will prove this important result of Archimedes, following a reasoning due to Luca Valerio, a mathematician highly respected by Galileo Galilei.



Is given a cylinder having a base radius r and height r, and inscribe in it a half sphere and a cone, as in the figure. Consider the cone and the solid obtained by subtracting the sphere from the cylinder (the *bowl* of Luca Valerio). Cutting these figures with a plane parallel to the base, you get two concentric sections: a circular crown A1 and a circle A2. The inner circle of the circular crown has for radius the cathetus of a right triangle whose hypotenuse is r and other cathetus h, so its area is:

$$A_i = p \cdot (r^2 - h^2)$$

The area of the outside circle of the crown is:

$$A_e = p \cdot r^2$$

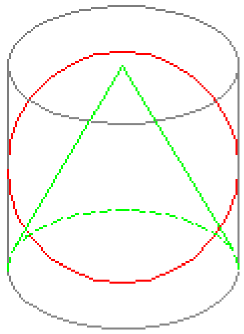
therefore the area of the crown is:

$$A_1 = A_e - A_i = p \cdot r^2 - p \cdot (r^2 - h^2) = p \cdot h^2$$

but this value coincides with that of the area of the section on the cone:

$$A_2 = p \cdot h^2$$

This equality is valid for all possible sectioning planes parallel to the base of the figures. Luca Valerio, considering these figures (the cone and the bowl) as composed by infinite "sheets" of infinitesimal thickness, generated by the sectioning planes, concludes that the two volumes, being composed of sheets of equal area, are equal (1). But the volume of the cone is a third of that of the cylinder (2). Then the volume of the half sphere is equal to two thirds of that of the cylinder. Doubling you get the statement of the theorem that you wanted to prove.



In a beautiful synthesis these results can be said, with reference to the schematic figure here at the left (3), in the following way:

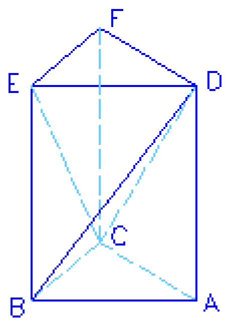
The three volumes of the cone, the sphere and the cylinder, are among them as the numbers 1,2 and 3.

Notes

(1) This is the method of "indivisibles", introduced by Bonaventura Cavalieri in 1635. We know now, from a discovery of scrolls in 1906, that this method, which anticipates the infinitesimal calculus, dating back to Archimedes. Cavalieri has just "rediscovered" it, after 1850 years!

(2) Euclid - The Elements - Book XII - Proposition 7:

Each prism having a triangular base divides into three pyramids equal to each other with triangular bases.



With reference to the figure, it is evident that:
ABDC=BEDC; BCED=CFED; CDFE=CADB

To extend to cylinder/cone is sufficient to consider the base circle as composed of triangles. A composition can be obtained from the figure that appears in the article "Measurement of the circle", taking the 6 equilateral triangles of the hexagon and adding the perimetral triangles that form successively the polygons of 12, 24, 48, etc.. sides, ad infinitum.

(3) It is said that Archimedes wanted engraved on his tombstone a similar figure, in memory of his great discovery. And is said that this wish was made perform from the consoles Marcello. Marcus Tullius Cicero said that at the time when it was Quaestor in Sicily, curiosity pushed him to look for the tomb of Archimedes, and being one day out of the door of Syracuse, among other graves he saw a column engraved with the figure of a sphere and a cylinder. Cleaned the site by the twigs he recognizes, from the inscriptions eroded by time, that this was precisely the tomb of Archimedes who was looking for.

http://upload.wikimedia.org/wikipedia/commons/c/c5/Bowl_02.gif

The surface of the sphere

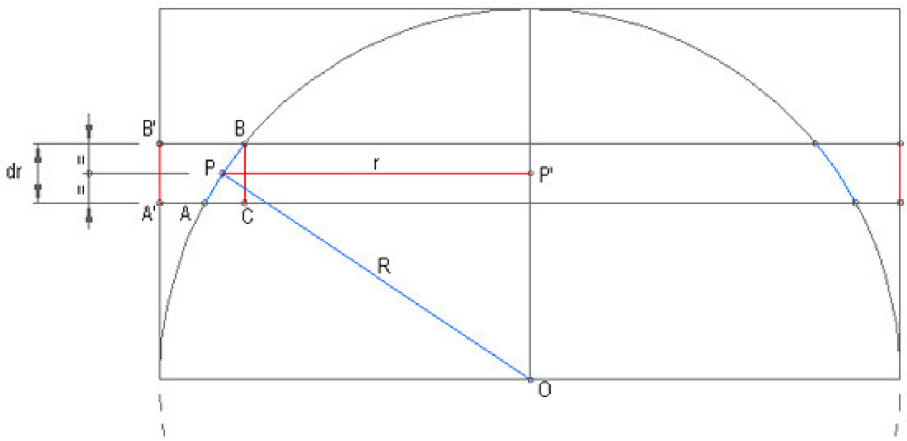
We shall prove the proposition that follows resorting again to the method of *indivisibles* of Cavalieri.

The surface of the sphere is equal to the lateral surface of a cylinder whose base is the great circle of the sphere and height the diameter of it .

This statement also says that the surface of the sphere is 4 times its maximum circle. In fact:

$$S = 2\pi r \cdot 2r = 4\pi r^2$$

Consider two generic horizontal planes distant from each other an infinitesimal dr, hat cut the surfaces of the sphere and the cylinder as in the figure.



We demonstrate that the lateral surface of the cylinder having elementary height A'B' = dr, is equal to the surface generated by revolution of the segment AB around the axis OP', namely:

$$dr \cdot 2\pi R = AB \cdot 2\pi r$$

$$dr \cdot R = AB \cdot r$$

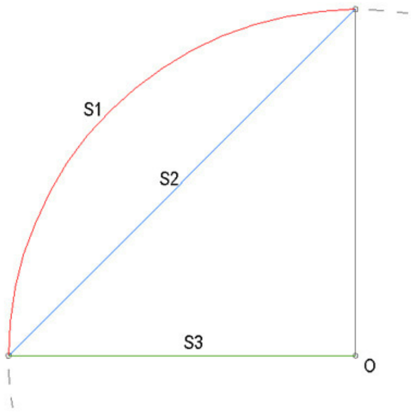
or:

$$dr / r = AB / R$$

But this proportion is evident in the figure, given the similarity of the triangles OPP' and ABC, formed by straight lines perpendicular to each other. Having taken the distances dr infinitely small, so rectify the arches APB, then, adding up all the elementary surfaces of the cylinder and the sphere, you get two equal surfaces.

From what has been shown , it follows an interesting proportion:
The lateral surface of a cone having base radius R and height R, is proportional mean between the lateral surface of a hemisphere of radius R and the surface of a circle of radius R, with a proportionality ratio $=\sqrt{2}$

We show for simplicity only the generatrices of the three figures (which are obtained by the revolution of these around the vertical axis through the point O):



We calculate S1/S2:

$$S1/S2 = 2\pi R^2 / \pi R^2\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$$

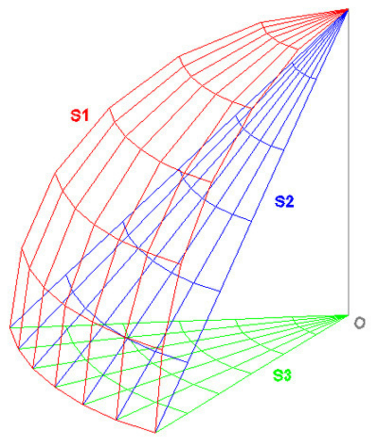
and then S2/S3:

$$S2/S3 = \pi R^2\sqrt{2} / \pi R^2 = \sqrt{2}$$

then the above said is valid, namely:

$$S1/S2 = S2/S3 = \sqrt{2}$$

Because we got rid of the 2π factor, the proportion is also true for any partial rotation, around their axis, of the three generatrices, that is, in the most general case of semi-slice. This leads to the classical form of the statement, applied to the "nails" or semi-slices: spherical, conical and flat:

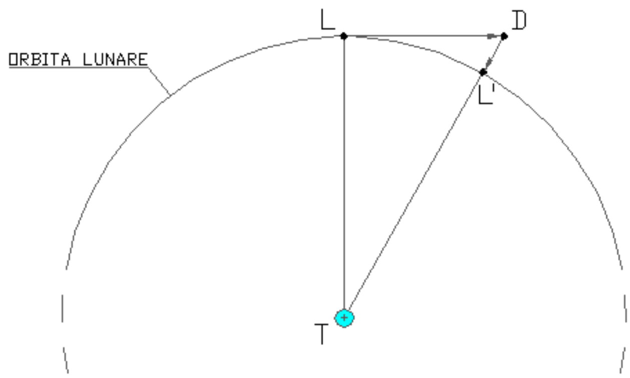


:

The Newton's apple

The force that causes the fall of an apple on the Earth is of the same nature as that which keeps the Moon in orbit around the Earth.

This great intuition of Newton, which led to the discovery of Universal Gravitation, will be proved here by comparing the result of a geometric-kinematic calculation to that of gravitational calculation.



You have:

$TL = 3,84*10^8 \ m$ Earth-Moon distance, in a first approximation

$p = 2,36*10^6 \ sec$ period of revolution of the Moon

$v = develop.\ orbit / p = 2* \pi * 3,84*10^8 / 2,36*10^6 =$
$= 1,02234*10^3 \ m/sec$ orbital velocity

$g = 9,8062 \ m/sec^2$ acceleration of gravity at Earth's surface

$R = 6,37*10^6 \ m$ radius of the Earth

$g' = g* R^2 / TL^2 = 2,6985*10^{-3} \ m/sec^2$ acceleration of gravity on lunar orbit

A - Geometric-kinematic calculation

In the absence of external forces, the Moon, with uniform rectilinear motion, would tread the distance LD in a second:

$LD = v*1 = 1022,34 \ m$

Instead the Moon remains in orbit at the point L', located on the line joining DT. We calculate geometrically the distance DL':

$DL' = (TL^2 + LD^2)^{1/2} - TL' = 1,3609*10^{-3} \ m$

B - Gravitational calculation

Due to the force of gravity exerted by the Earth, Moon falls from point D to point L' accomplishing in a second the space DL' that is:

$$DL' = ½* g'* t^2 = ½* 2,6985*10^{-3} *1 = 1,3492*10^{-3} m$$

As you can see, the mechanical calculation provides for DL' the same value (unless the made approximations) of the geometric calculation. So, the force that keeps the moon in orbit is the Earth's gravity.

Let's see how you can use a spreadsheet to get the value of the Earth-Moon distance, at which you get two equal results for the distance DL'.
Placed in the head all the variables of our discussion, in the first row you enter data and formulas used above, as follows:

TL	p	v	LD	DL'	g	R	g'	DL'
384000000	2375000	=2*PI.GRECO()*A2/B2	=+C2	=+(A2^2+D2^2)^0,5-A2	9,8062	6367472	=+(G2^2/A2^2)*F2	=+H2/2

You are copying down the content of all the cells of the first row, excluding the TL value. So you does vary, in the first column, the value of TL to obtain in columns 5 and 9, the equality of the values of DL ':

TL	p	v	LD	DL'	g	R	g'	DL'
384000000	2375000	1015,8919	1015,8919	0,00134379	9,8062	6367472	0,002696	0,00134816
384050000	2375000	1016,0241	1016,0241	0,00134397	9,8062	6367472	0,002696	0,00134781
384100000	2375000	1016,1564	1016,1564	0,00134414	9,8062	6367472	0,002695	0,00134746
384150000	2375000	1016,2887	1016,2887	0,00134432	9,8062	6367472	0,002694	0,00134711
384200000	2375000	1016,4210	1016,4210	0,00134450	9,8062	6367472	0,002694	0,00134676
384250000	2375000	1016,5532	1016,5532	0,00134468	9,8062	6367472	0,002693	0,00134641
384300000	2375000	1016,6855	1016,6855	0,00134486	9,8062	6367472	0,002692	0,00134606
384350000	2375000	1016,8178	1016,8178	0,00134504	9,8062	6367472	0,002691	0,00134571
384400000	2375000	1016,9501	1016,9501	0,00134516	9,8062	6367472	0,002691	0,00134536
384402000	2375000	1016,9554	1016,9554	0,00134522	9,8062	6367472	0,002691	0,00134534
384404000	2375000	1016,9607	1016,9607	0,00134522	9,8062	6367472	0,002691	0,00134533
384406000	2375000	1016,9659	1016,9659	0,00134522	9,8062	6367472	0,002691	0,00134532
384408000	2375000	1016,9712	1016,9712	0,00134522	9,8062	6367472	0,002691	0,00134530
384410000	2375000	1016,9765	1016,9765	0,00134522	9,8062	6367472	0,002691	0,00134529
384412000	2375000	1016,9818	1016,9818	0,00134522	9,8062	6367472	0,002691	0,00134527
384414000	2375000	1016,9871	1016,9871	0,00134528	9,8062	6367472	0,002691	0,00134526
384416000	2375000	1016,9924	1016,9924	0,00134528	9,8062	6367472	0,002690	0,00134525
384418000	2375000	1016,9977	1016,9977	0,00134522	9,8062	6367472	0,002690	0,00134523

The found value of $3,8441*10^8$ *m* for the Earth-Moon distance, agrees, as it should be, with the results of the most recent laser measurements, which provide an average value of *384.400 Km*.

The coconut problem

Here's an example of how you can use a spreadsheet to find the solution to a complicated math problem presented at the beginning of the twentieth century and known as the "coconut problem".

Five men are shipwrecked on an island. They do not find anything to eat except many coconuts; also find a monkey. They decide to divide the coconuts into five equal parts, leaving the remains to the monkey. In the middle of the night, one of the castaways suddenly feels hungry and decides to take his share of coconuts. In doing so discovers that dividing by five the number of coconuts, it has as a remainder 1; so he gives a coconut to the monkey, takes his fifth and puts the rest of the coconuts back into a pile . Shortly after he wakes up a second castaway and does exactly the same thing: it gives the monkey a coconut, takes his fifth and puts the remaining coconuts into a pile . The other three do the same thing.

The next morning all rise, divide what remains of coconuts in equal parts and again there is one left for the monkey.

How many coconuts were in the original pile?

The calculations can be organized on a spreadsheet as follows.

In the first field is a list of the possible integers giving the hypothetical final amount of coconuts. In the five subsequent fields, we introduce the formulas to get, *backlinks*, the amount of nuts that each castaway finds at night at his disposal. The sixth field will then contain the initial amount of coconuts.

Let us now see what are the values to be introduced in the first field. By analyzing the actions of the morning, these numbers must be divisible by 5, increased by 1; also, so that by them can rebuild the previous amount adding ¼, must be numbers divisible by 4. To achieve this, we use the spreadsheet as follows:

n	n+1	(n+1)/4
5	6	1,5
10	11	2,75
15	16	4
20	21	5,25
25	26	6,5
30	31	7,75
35	36	9
40	41	10,25
45	46	11,5
50	51	12,75
55	56	14
ecc...		

The sequence of numbers to be entered in the first field is given by the numbers highlighted in yellow in the second column. The first number is 16 and the others are obtained by adding successively to the first the number 20 (which is the smallest number divisible by 5 and 4). To rebuild the quantities in the subsequent fields simply increase by 1/4 the amount of the previous field and add 1. Introduced the formulas in each field of the first record and copying at the bottom, you have the following table:

MONTE FINALE	5° MONTE	4° MONTE	3° MONTE	2° MONTE	MONTE INIZIALE
16	21	27,25	35,0625	44,82813	57,03516
36	46	58,5	74,125	93,65625	118,0703
56	71	89,75	113,1875	142,4844	179,1055
76	96	121	152,25	191,3125	240,1406
96	121	152,25	191,3125	240,1406	301,1758
116	146	183,5	230,375	288,9688	362,2109
136	171	214,75	269,4375	337,7969	423,2461
156	196	246	308,5	386,625	484,2813
176	221	277,25	347,5625	435,4531	545,3164
196	246	308,5	386,625	484,2813	606,3516
216	271	339,75	425,6875	533,1094	667,3867
236	296	371	464,75	581,9375	728,4219
256	321	402,25	503,8125	630,7656	789,457
276	346	433,5	542,875	679,5938	850,4922
296	371	464,75	581,9375	728,4219	911,5273
316	396	496	621	777,25	972,5625
336	421	527,25	660,0625	826,0781	1033,598
356	446	558,5	699,125	874,9063	1094,633
376	471	589,75	738,1875	923,7344	1155,668
396	496	621	777,25	972,5625	1216,703
416	521	652,25	816,3125	1021,391	1277,738
:	:	:	:	:	:

Of course, the solution that we try you will find at the first record containing a sextuple of integers:

.	.	.	.	.	.
5096	6371	7964,75	9956,938	12447,17	15559,96
5116	6396	7996	9996	12496	**15621**
5136	6421	8027,25	10035,06	12544,83	15682,04
.	.	.	.	.	.

So at the beginning the coconuts were 15621. Continuing the search for sextuplets in the entire spreadsheet, we find a second:

.	.	.	.	.	.
10216	12771	15964,75	19956,94	24947,17	31184,96
10236	12796	15996	19996	24996	**31246**
10256	12821	16027,25	20035,06	25044,83	31307,04
.	.	.	.	.	.

and one meets others, all spaced in the first field of 5120 units:

MONTE FINALE	5° MONTE	4° MONTE	3° MONTE	2° MONTE	MONTE INIZIALE
5116	6396	7996	9996	12496	**15621**
10236	12796	15996	19996	24996	**31246**
15356	19196	23996	29996	37496	**46871**
20476	25596	31996	39996	49996	**62496**
25596	31996	39996	49996	62496	**78121**
30716	38396	47996	59996	74996	**93746**
35836	44796	55996	69996	87496	**109371**
40956	51196	63996	79996	99996	**124996**
46076	57596	71996	89996	112496	**140621**
51196	63996	79996	99996	124996	**156246**
56316	70396	87996	109996	137496	**171871**
:	:	:	:	:	:

You could therefore be said, by the induction principle, that there are infinite solutions to this problem.

Observations

1. The final table is interesting for some curious regularities present in it, for example: the sequence of the values of the third pile is obtained from the first value by adding successively the number 10000, and again, each record is obtained from the previous one by adding the values of the first record + 4.

2. The Nobel Prize for Physics Paul Dirac, gave a solution to this problem mathematically correct, whose record in a table constructed as above, would be the following:

MONTE FINALE	5° MONTE	4° MONTE	3° MONTE	2° MONTE	MONTE INIZIALE
-4	-4	-4	-4	-4	-4

Each castaway find -4 coconuts, by subtracting one obtains -5, takes one fifth leaving so -4 at the next castaway, and at the final pile. This solution is clearly impossible, but if we apply to the table the rule discovered in the first observation, according to which each record would be obtained from the previous by adding the values of the first record + 4, we get another solution, this time too realistic, in which the castaways are destined to die of hunger, having not found coconuts on the island. You can still observe that the Dirac's sextuple can be achieved by applying the above rule at the first record in the table of solutions, but on the contrary, i.e. by subtracting the values of the first record + 4, this confirms the correctness of the Dirac's mathematical solution. It is said that this negative solution has had an influence on the thought of Dirac, which then would have introduced the concept of antimatter.

3. Continuing with the observations on the general spreadsheet, you can still note that the sequences of integers, results neatly grow at records n. 1, 4, 16, 64, 256 (the powers of the number 4):

REC.	MONTE FINALE	1 5° MONTE	2 4° MONTE	3 3° MONTE	4 2° MONTE	5 1° MONTE
1	16	**21**				
4	76	96	**121**			
16	316	396	496	**621**		
64	1276	1596	1996	2496	**3121**	
256	5116	6396	7996	9996	12496	**15621**

It glimpses the possibility of obtaining from this table a formula to calculate, for each record, the number of coconuts in correspondence of the various piles, i.e. the mathematical solution of our problem.

Denoting with n the number of the pile (in green) and employing appropriately the numbers 4 and 5, characteristics of the problem, we find that:

The values of the "final pile", denoted by $Mf\,(r)$, where r is the position of the record in the previous summary table, can be calculated as follows:

$$M_f(r) = 5 \bullet 4^r - 4$$

for example, at $r=3$:

$$M_f(3) = 5 \bullet 4^3 - 4 = 316$$

The values of the nth pile to the record r, denoted by $M(n,r)$, are obtained by repeating n times the addition of 1/4 + 1, i.e. in the following typical manner, valid for $n = 3$, $r = 3$:

$$M(3,3) = ((M_f(3) \bullet 5/4 + 1) \bullet 5/4 + 1) \bullet 5/4 + 1$$
$$= M_f(3) \bullet (5/4)^3 + (5/4)^2 + 5/4 + 1 =$$
$$= 316 \bullet (5/4)^3 + (5/4)^2 + 5/4 + 1 = 621$$

Generalizing, we obtain:

$$M(n,r) = M_f(r) \bullet (5/4)^n + (5/4)^{n-1} + \cdots + (5/4)^{n-n}$$

that is:

$$M(n,r) = (5 \bullet 4^r - 4) \bullet (5/4)^n + \sum_{k=0}^{n-1} (5/4)^k \qquad (1)$$

which is finally the formula that were looking for.

The Binet formula for the Fibonacci numbers

The Fibonacci sequence is a sequence of natural integers defined as follows:

$$F_n = F_{n-2} + F_{n-1} \qquad \forall\, n \geq 2$$
$$F_0 = 0 \qquad (1)$$
$$F_1 = 1$$

The development of such a sequence is:

$$(F_n) = \{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\ldots\} \qquad (2)$$

The main property of the succession is one for which:

$$\lim_{n\to\infty} F_n / F_{n-1} = \Phi \quad \text{(golden section)} \qquad (3)$$

where:

$$\Phi = (1+\sqrt{5})/2 = 1{,}6180339... \qquad (4)$$

Of course, the relationship between F_n and its subsequent tend to the reciprocal of the golden section:

$$\lim_{n\to\infty} F_{n-1} / F_n = \phi = (1-\sqrt{5})/2 = -0{,}6180339... \qquad (5)$$

Each element of the Fibonacci sequence is obtained from the Binet formula:

$$F_n = (\Phi^n - \phi^n)/\sqrt{5} \qquad \forall n \in \mathrm{N}_0 \qquad (6)$$

To show the validity of the Binet formula, we consider the difference:

$$\Delta_n = \Phi_n - \phi_n = ((1+\sqrt{5})/2)^n - ((1-\sqrt{5})/2)^n \qquad (7)$$

Calculating this difference in correspondence of the first natural numbers is obtained:

$$\Delta_0 = \Phi^0 - \phi^0 = 0$$

$$\Delta_1 = \Phi^1 - \phi^1 = \sqrt{5} \qquad (8)$$

$$\Delta_2 = \Phi^2 - \phi^2 = \sqrt{5}$$

You could say that the two previous results, equals among them and having value $\sqrt{5}$ can be obtained only using the golden number and its reciprocal. Indeed, from the following system:

$$x - y = \sqrt{5}$$

$$x^2 - y^2 = \sqrt{5}$$

you get the *unique* solution:

$$x = (1+\sqrt{5})/2 = \Phi = 1{,}6180339...$$

$$y = (1-\sqrt{5})/2 = \phi = -0{,}6180339...$$

Therefore $\sqrt{5}$ seems to be an excellent candidate for the construction of the succession:

$$(\Delta_n) = (F_n)\sqrt{5} = \{0, 1\sqrt{5}, 1\sqrt{5}, 2\sqrt{5}, 3\sqrt{5}, 5\sqrt{5}, 8\sqrt{5}, 13\sqrt{5}, ...\}$$

Therefore $\sqrt{5}$ seems to be an excellent candidate for the construction of the succession:

$$\Delta_3 = \Phi^3 - \phi^3 = 2\sqrt{5}$$

$$\Delta_4 = \Phi^4 - \phi^4 = 3\sqrt{5} \qquad (8)$$

$$\Delta_5 = \Phi^5 - \phi^5 = 5\sqrt{5}$$

The succession of the results continues, by mathematical induction, with values that are all integer multiples of $\sqrt{5}$ reproducing exactly the Fibonacci sequence. It has namely:

$$\Delta_n = \Phi^n - \phi^n = F_n\sqrt{5} \qquad \forall n \in N_0 \qquad (9)$$

and the validity of the Binet formula (6) is thus verified.

The Binet formula appears surprising, for the fact that from it, made up of irrational elements, are obtained by varying n only natural numbers. In fact you can see, performing calculations (8) to find Δ_n, like all the roots cancel each other, leaving the natural numbers as the final result.

We now perform a check of the above using a spreadsheet:

n	F_n	F_n/F_{n-1}	F_{n-1}/F_n	Φ	φ	$\Delta_n=\Phi^n-\varphi^n$	$\frac{\Phi^n-\varphi^n}{5\wedge 0,5}$
0	0			1,618034	-0,618034	0,0000	0
1	1		0,0000000000000	1,618034	-0,618034	2,2361	1
2	1	1,0000000000000	1,0000000000000	1,618034	-0,618034	2,2361	1
3	2	2,0000000000000	0,5000000000000	1,618034	-0,618034	4,4721	2
4	3	1,5000000000000	0,6666666666667	1,618034	-0,618034	6,7082	3
5	5	1,6666666666667	0,6000000000000	1,618034	-0,618034	11,1803	5
6	8	1,6000000000000	0,6250000000000	1,618034	-0,618034	17,8885	8
7	13	1,6250000000000	0,6153846153846	1,618034	-0,618034	29,0689	13
8	21	1,6153846153846	0,6190476190476	1,618034	-0,618034	46,9574	21
9	34	1,6190476190476	0,6176470588235	1,618034	-0,618034	76,0263	34
10	55	1,6176470588235	0,6181818181818	1,618034	-0,618034	122,9837	55
11	89	1,6181818181818	0,6179775280899	1,618034	-0,618034	199,0100	89
12	144	1,6179775280899	0,6180555555556	1,618034	-0,618034	321,9938	144
13	233	1,6180555555556	0,6180257510730	1,618034	-0,618034	521,0038	233
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

The first column of the sheet contains the sequence of natural numbers.

The second column shows the corresponding Fibonacci sequence.

In the third column we see how the terms of the sequence (3) tends to the golden section Φ, as n tends to infinity.

As above in the fourth column for the terms of the sequence (4), tending to the number ϕ.

The fifth and sixth columns are used to calculate the seventh.

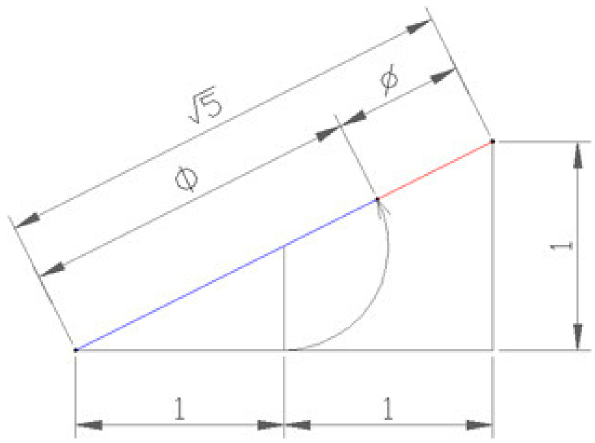
The seventh column contains the succession of the values obtained with the above calculations (8).

The eighth column lists the values obtained by the Binet formula (6) by varying n, which correspond exactly to the values of the second column, i.e. the Fibonacci sequence.

Patterns

As usual, also this time I searched geometric patterns, finding for the formulas (8) that calculate the Δ_n differences, the following curiosity.

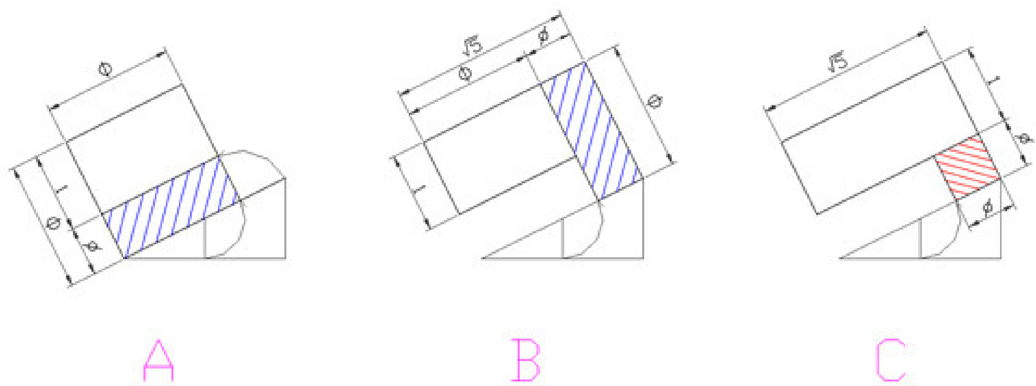
1 - For Δ_1 we have, in the R space:



On the hypotenuse of the right triangle in the figure, long $\sqrt{5}$, one can construct the segments of length Φ e ϕ, as shown. You can see how:

$$\Delta_1 = \Phi^1 - \phi^1 = \sqrt{5}\,u \text{ (linear units)}$$

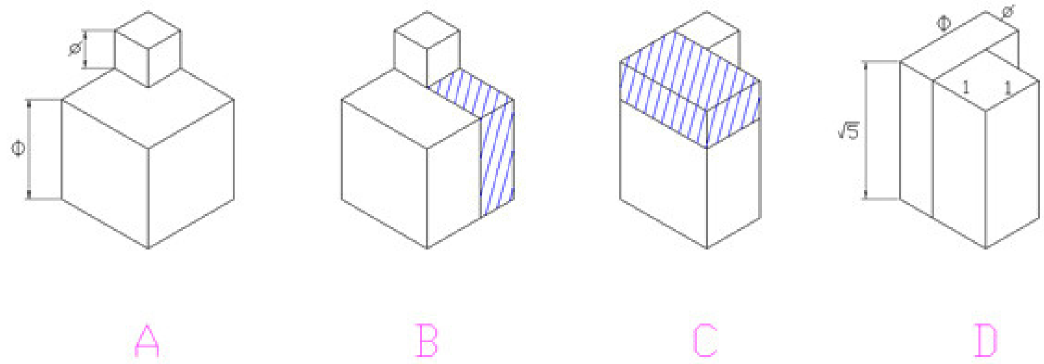
2 - For Δ_2 we have, in the R^2 space:



The square with side Φ (A) can be transformed into the L figure of equal surface (B). In the figure C we see as:

$$\Delta_2 = \Phi^2 - \phi^2 = \sqrt{5}\,u^2$$

3 - For Δ_3 we have, in the R^3 space:



With successive transformations is obtained, from the solid in figure A, the solid in figure C of equal volume. In figure D we see as such a solid can be decomposed into the sum of two parallelepipeds of volume equal to $\sqrt{5}\,u^3$, being: $\Phi - \phi = 1$ and $\Phi * \phi = 1$. Then you have:

$$\Delta_3 = \Phi^3 - \phi^3 = 2\sqrt{5}\,u^3$$

Euclid’s algorithm with AutoCad

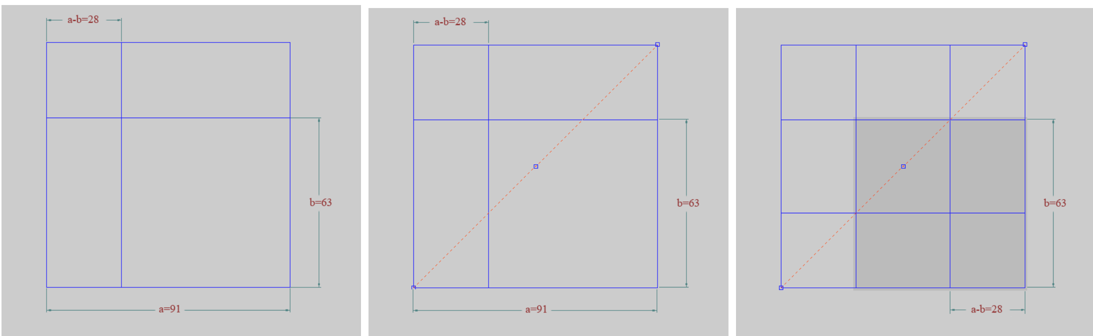
The *Euclidean algorithm*, in its simplest form, is a method for calculating the *greatest common divisor* (GCD) of two positive integers. Euclid described this method, based on "successive subtractions”, in his book “*Elements*”. Starting from a pair of positive integers, he subsequently form new pairs, each constituted by the smallest number of the previous pair and the difference between the numbers of the same pair.

The algorithm is based on the following property:

If two numbers a and b are divisible by a third number k, then also their difference is divisible by k.

Proof
Suppose a > b, then:
a = k ⋅ m
b = k ⋅ n
a-b = k ⋅ (m-n)
One can therefore say, in the particular case in which k = GCD(a, b):
GCD(a,b) = GCD((a-b),b)

In the animation linked below, is shown a graphic application of the Euclidean algorithm to get, using AutoCAD, the GCD of two line segments, whose entire lengths are a and b. In it, the "successive subtractions" are obtained by flipping squares around their diagonal axis, as follows:



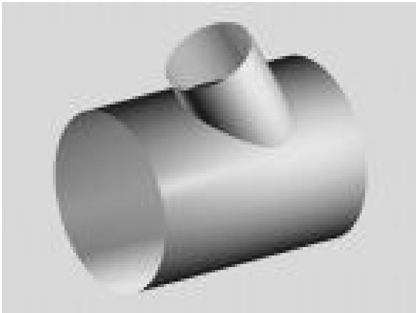
However, this is an implementation of low practical interest, since there are more efficient, still derived from Euclid, based on "successive divisions."

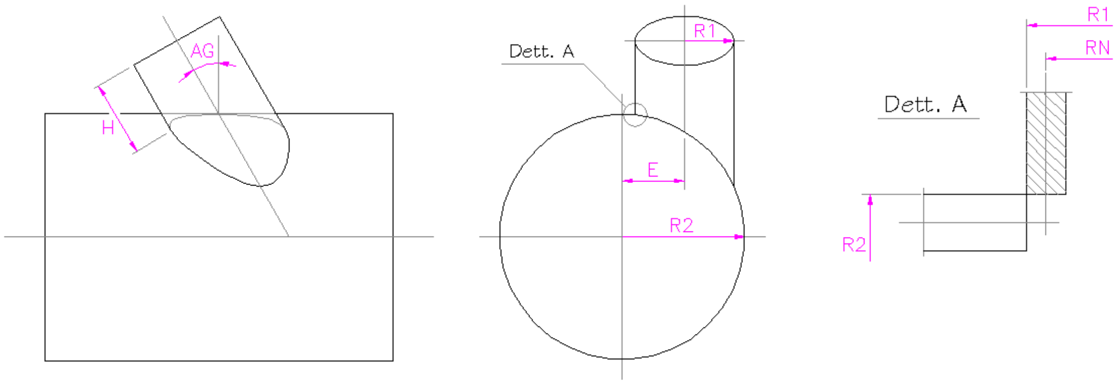
Link: http://youtu.be/Bp1DT39t3cs

A practical use of AutoCAD

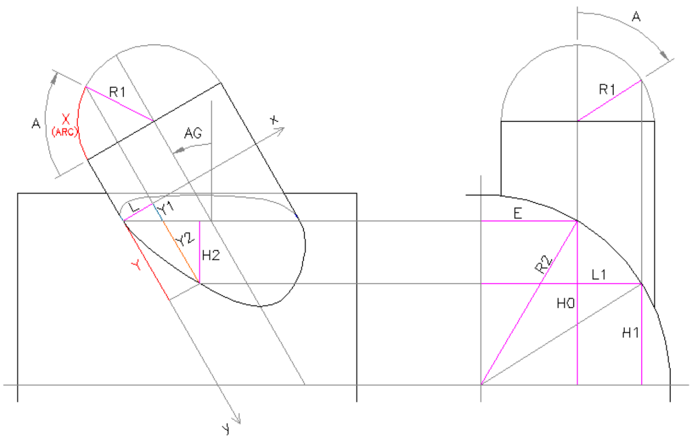
Here I will present a graphical implementation of AutoCAD that, unlike the previous, has a practical interest, being employable in the workshops to get plan developments of 3D surfaces that mostly occur in the construction typology of Pressure Vessels and Piping components made from steel plates.

The case is that of the plan development of an oblique intersection between cylinders, shown in the figure:





The formulas to create the AutoLISP program are obtained from the following figure:



$$X = R1 \times A$$

$$L = R1 - R1 \times \cos A$$

$$Y1 = L \times \tan AG$$

$$H0 = \sqrt{R2^2 - E^2}$$

$$L1 = E + R1 \times \sin A$$

$$H1 = \sqrt{R2^2 - L1^2}$$

$$H2 = H0 - H1$$

$$Y2 = H2 / \cos AG$$

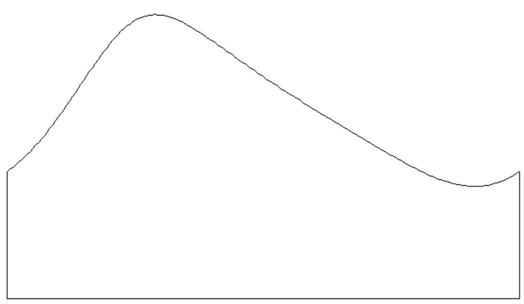
$$Y = Y1 + Y2$$

The program listing, written with the text editor of AutoCAD, is this:

```
(DEFUN C:BOCC2 ()
   (GRAPHSCR)
      (SETQ AG (GETREAL "\nIMMETTERE ANGOLO AG: ")
            R1 (GETREAL "\nIMMETTERE RAGGIO R1: ")
            RN (GETREAL "\nIMMETTERE RAGGIO RN: ")
            R2 (GETREAL "\nIMMETTERE RAGGIO R2: ")
            E (GETREAL "\nIMMETTERE ECCENTRICITA' E: ")
            H (GETREAL "\nIMMETTERE ALTEZZA H: ")
            P1 (GETPOINT "\nPUNTO INIZIALE: ")
      )
         (SETQ P1 (LIST 0 0))
         (SETQ A (/ PI 90))
            (REPEAT 180
               (SETQ L (- R1 (* R1 (COS A)))
                     Y1 (* L (/ (SIN (* (/ PI 180) AG)) (COS (* (/ PI 180)
AG))))
                     H0 (SQRT (- (* R2 R2) (EXPT E 2)))
                     L1 (+ E (* R1 (SIN A)))
                     H1 (SQRT (- (* R2 R2) (* L1 L1)))
                     H2 (- H0 H1)
                     Y2 (/ H2 (COS (* (/ PI 180) AG)))
                     Y (+ Y1 Y2)
                     K (/ RN R1)
                     X (* (* K R1) A) P2 (LIST X Y)
               )
                  (COMMAND "LINE" P1 P2 "")
                  (SETQ P1 P2)
                  (SETQ A (+ A (/ PI 90)))
            )
                  (SETQ P3 (LIST (CAR P2) (- 0 H)))
                  (SETQ P4 (LIST 0 (- 0 H)))
                  (SETQ P5 (LIST 0 (CADR P2)))
                  (COMMAND "LINE" P2 P3 P4 P5 "")
                  (COMMAND "ZOOM" "ALL" "")
                  (PRINC)
)
```

This program saved in the file: BOCC2.LSP, can be used as an AutoCAD command, uploading it from "Tools" menu, with "Load Application" command (AutoCad 14).

You get the following graphic output, used to program an automatic plant for thermal cut of steel plates.



For insights into this topic you can visit my website:

https://sites.google.com/site/quadernitecnici/home

where they presented 25 different AutoLISP programs freely downloadable.

The girl on the Round lake

I propose now a problem of formal logic, appeared in the magazine "Scientific American" many years ago. It is a problem not easy to solve, to be addressed using logic and math high school type.

A very pretty girl is on holiday on the Round lake, so called because it has a perfectly circular shape with radius r. One day the girl, while walking along the shore of the lake, is approached by a brute who tries to rape her. The girl manages to wriggle out and began to run around the lake, chased by brute. Running away, he sees a boat on the shore, so he enters inside and goes to the center of the lake to save. The brute runs with a speed exactly 4 times greater than the speed of the girl in the lake.
Given that:

- The brute can not swim and cannot do nothing but run in circles;
- The girl is always aware of their location coordinates;

there arises the question: how does the girl to save himself?

Solution
We denote by A and B positions of the girl and the brute, respectively.
Let us examine a hypothetical situation in which the girl and the brute will encounter on the lake shore, at a point X, at the same time. For this to happen, the girl A should be, at some point in his escape, at a position opposite to B (Figure 1) and far from the shore of a stretch:

d = $(\pi / 4) \cdot r$

The girl will be saved if:

$(3/4) \cdot r <$ d $< (\pi / 4) \cdot r$ (1)

But how can the girl, from the center of the lake, reaching the position shown in Figure 1?
Let us examine Figure 2, in which is shown a concentric circular path passing through the position A of Figure 1. If the girl, starting from the center, arrives at any point of this path, can follow it so as to reach, at some time, the salvation position of Figure 1. In fact, as one can easily see, on this circular path the angular velocity of the girl turns out to be slightly higher than that of the brute. So, reached the opposition with B, the girl can continue along the radius and save.

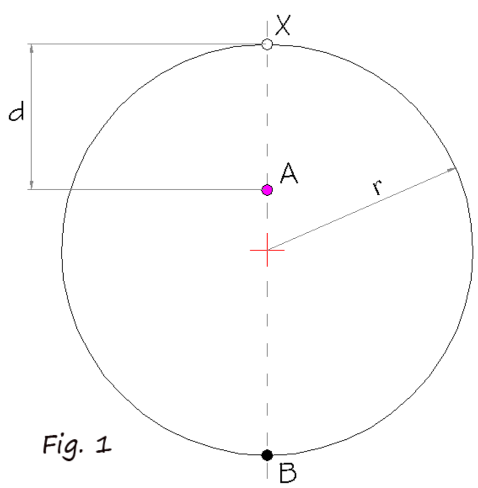


Fig. 1

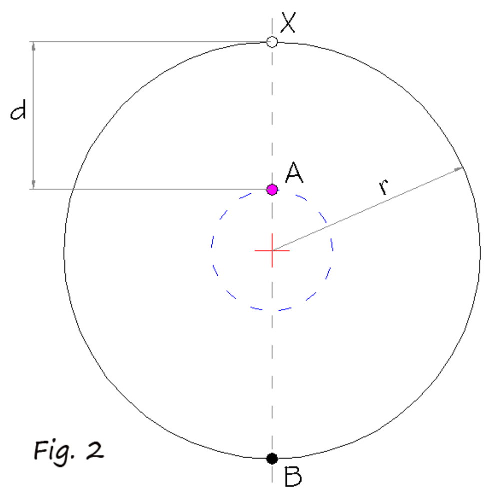


Fig. 2

The relationship (1) describes a thin circular crown, inside which all the concentric circular paths are valid. Are excluded the extremes of the range (= signs), which give:

- the smallest: an angular speed equal to that of the brute, and therefore the impossibility to achieve the opposition that leads to salvation;
- the largest: the run-up to a fateful rendezvous.

Fermat's last theorem: a challenge still open

The history of this theorem is very well known, however we give here a brief summary of it.

It is said that the mathematician Pierre de Fermat, struggling with the reading of an edition of the Diophantu's Arithmetica, has annotated on its margin the following statement:
« it is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second, into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain »

In doing so, maybe without realizing it, he had posed a question to which the ancient Greek mathematicians had not thought, formulating, in a way that astonishes by its simplicity, the most difficult problem that has ever existed.

In later centuries many great mathematicians tried the "lost" proof of this theorem. Euler demonstrated the impossibility of integer solutions for n = 3. Other mathematicians, such as Legendre and Lejeune-Dirichlet, proved independently the case n = 5. But none of them has ever reached a conclusion, ie a rigorous mathematical proof of Fermat's conjecture valid for any n.

Finally, in 1994, Andrew Wiles of Princeton University, with the aid of Richard Taylor, gave a demonstration of Fermat's last theorem, published in 1995 in the Annals of Mathematics.

Years before, two Japanese mathematicians, Yutaka and Shimura, had conjectured (without proving it) a deep interconnection between "Elliptic Curves" and "Modular Forms", two completely different fields of mathematics. This conjecture states that any elliptical curve is modular, and may be associated with a unique modular form. Later, other researchers proved that, if Fermat's conjecture were not true, there would be a solution (a, b, c) for n > 2, with which it would be possible to create a non-modular elliptic curve, against the conjecture of Yutaka and Shimura. Then, it was enough to prove the conjecture of Yutaka and Shimura to solve Fermat's last theorem. To do this, Wiles has employed seven years, most of them in complete isolation, making a titanic effort to converge all the latest techniques of number theory toward his proof of Fermat's theorem.

So, in addition to the work of other researchers in different fields of mathematics of the XX century, Wiles uses techniques and tools that were unknown at the time of Fermat. The end result is a big job (over 300 pages) written in terms very difficult, even for most of today's mathematicians.

Conversely, if it exists and is valid, the proof of Fermat should be much more concise and elementary, worthy of the beauty of his statement (marvelous), written in the terms used by mathematicians of the XVII century. Therefore, according to this hypothesis, the search for the lost proof would be considered a challenge still open.

However, many mathematicians are more likely to believe that Fermat was referring, in his note in the margins of Diophantus, to the existence of an his proof, initially deemed to be correct, but then abandoned or put aside due to afterthoughts, as has happened to many later mathematicians who have grappled with the same problem. A proof of this would be the existence of a partial job on the impossibility of integer solutions for n = 4, published by Fermat in a later period, which shows that for him the search was not yet complete.

Recently, new proofs of Fermat's theorem have been proposed to the attention of the international scientific community, a sign that the challenge is still open and many mathematicians continue the search for the simplest proof that Fermat could have in mind.

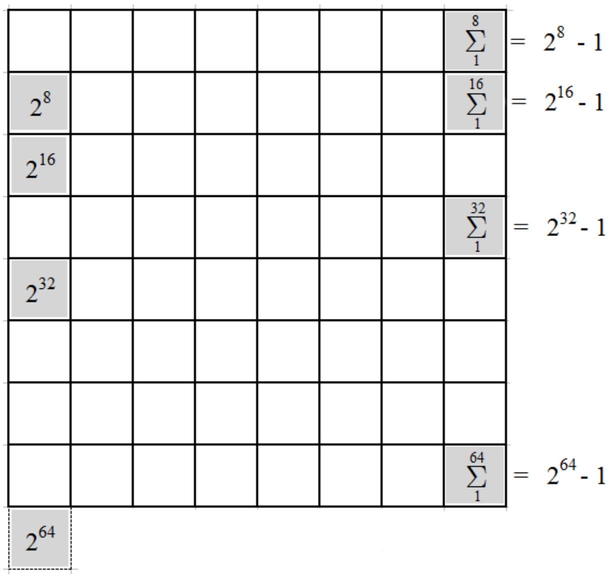
Chessboard problems

1 - The legend of Sissa

It is an Indian legend about the birth of chess, according to which an Indian prince, wanting to reward his Brahmin Sissa for the invention of the game of chess, listened a seemingly modest request. Would have to pay a grain of wheat on the first square of the chessboard, two on the second, four on the third, and so on, always doubling, up to cover the entire board. The prince promised prize, but then he realized he did not have enough grain to maintain its promise. Sissa definitely not had what he asked, as the aforesaid sum turns out to be a very high number:

18446744073709551615 (about 1000 billion tons of grain)

To find this number Leonardo Fibonacci (Liber Abaci-1202) calculates the result of the first line (8 boxes) that is 255, less than one unit of the next number $256 = 2^8$, the first of the second row. Then, doubling three times the number of boxes to be counted, notes that:



$(2^8)^2 = 2^{16} = 65536$ sum of the first 16 boxes plus 1

$(2^{16})^2 = 2^{32} = 4294967296$ sum of the first 32 boxes plus 1

$(2^{32})^2 = 2^{64} = 18446744073709551616$ sum of all the 64 boxes plus 1

Therefore, the number of grains of wheat, that we denote by S, is:

$$S = 2^{64} - 1 = 18446744073709551615$$

The same result is obtained from the geometric progression with common ratio q:

$S_n = 1 + q + q^2 + ... + q^{n-1}$

$S_n \cdot q = q + q^2 + q^3 + ... + q^n$

$S_n \cdot q - S_n = q^n - 1$

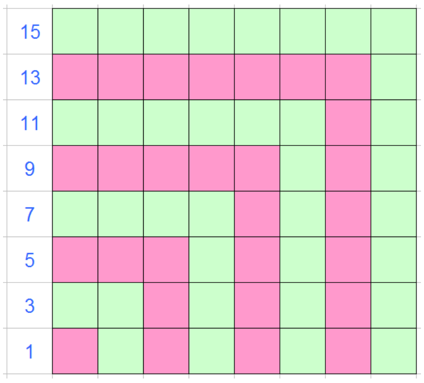
In our case q = 2, then:

$S_n \cdot 2 - S_n = 2^n - 1$

$S_{64} = 2^{64} - 1$

2 - Sum of the first 'n' odd numbers

In my opinion, the most immediate way to find the formula that calculates the sum of the first n odd numbers is the one that uses a chessboard, or any square grid.



Taking as a unit a square of the chessboard and highlighting the sequence of odd numbers as in the figure, we immediately see that:

$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64 = 8^2$

and we can write, by induction:

$$\sum_{k=1}^{n} (2n-1) = n^2$$

3 - How many squares you see on a chessboard?

The squares with unit side are $8^2 = 64$.

The squares with side greater than 1 are neatly counted moving on the rows (or columns) one column (or row) at a time. You get:

7^2 squares with side 2
6^2 squares with side 3
5^2 squares with side 4
4^2 squares with side 5
3^2 squares with side 6
2^2 squares with side 7
1 square with side 8

Therefore, the total number of visible squares is the sum of the squares of the first 8 natural numbers:

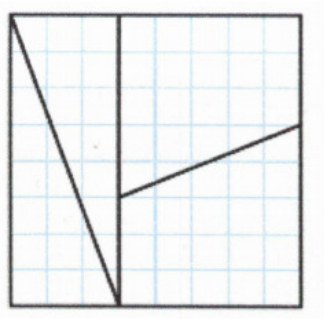
$1 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 = 204$

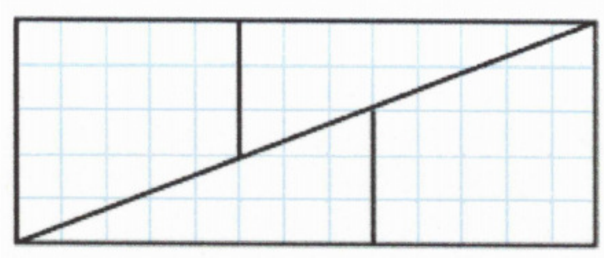
In general, in a n x n square board, visible squares are given by the square pyramidal number (him again!):

$$P_n = \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}$$

4 - The riddle of the broken chessboard

We break a chessboard 8 x 8 in four parts, as shown in the first figure, and reassemble it as in the second figure.



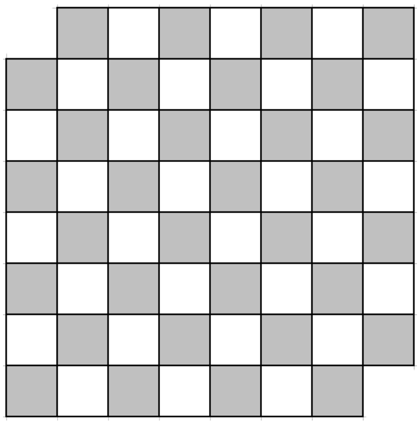


Each figure consists of two right triangles having base 8 and height 3 and two right trapezoids having height 5 and parallel bases long 5 and 3. But the square on the left contains 64 small squares, while the rectangle contains 13 x 5 = 65. There is a square in more, how come?

The riddle dissolves easily. Just to compare, in rectangular shape, the inclinations of the hypotenuse of a triangle and of the oblique side of the adjacent trapezoid. In fact, the hypotenuse has a slope equal to 3/8 = 0.375, while the oblique side has a slope of 2/5 = 0.4. Being inclined differently the two segments may not belong, as it seems, to the diagonal of the rectangle that is a straight line. These subtle differences between the actual figure and the apparent form the square in more.
There is another way to solve the riddle. One can find the *exact* difference of a square by calculating the areas of the triangles with the Pick's theorem (the *lattice point* on the hypotenuse appears only in the second figure).

5 - The mutilated chessboard

In a chessboard were removed the two corner squares, as shown. You take 31 rectangular tiles such that each tile covers exactly two squares.
Question: Is it possible to arrange 31 tiles to cover the entire board with them?



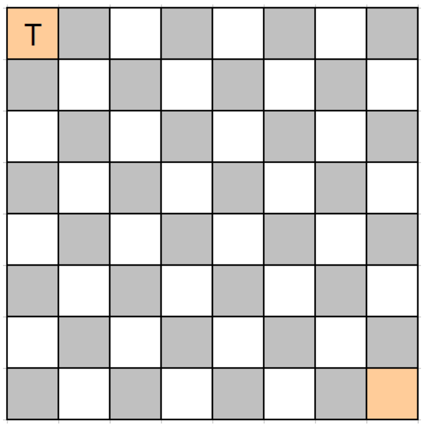
The problem is solved with the following reasoning:

- The board is made up of 30 white squares and 32 black.
- Two adjacent squares always have different colors.
- Then, in any manner are arranged, the first 30 tiles covered 30 black and 30 white squares.
- The two remaining squares will always be black, so they cannot both be covered by the remaining tile.

Therefore, any combination of tiles will not be able to cover the mutilated chessboard.

6 - The step of the tower

You can reach with a tower, starting from the box in the upper left, the box in the lower right corner, passing only once on all 64 squares of the chessboard?



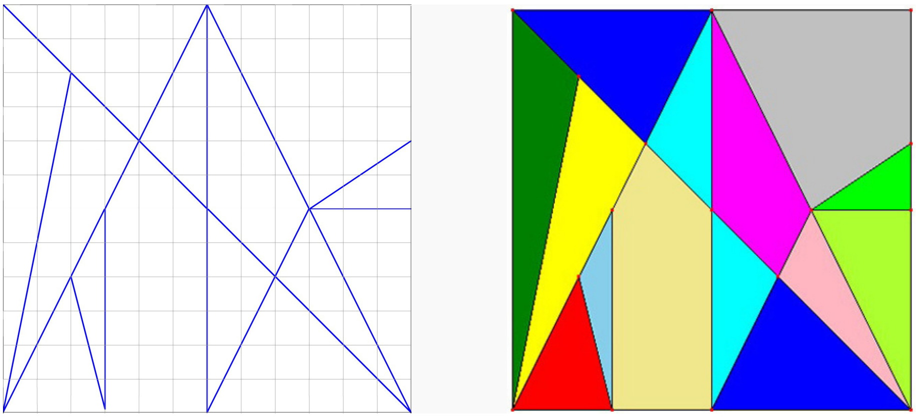
The problem is solved with the following reasoning:

- Each elementary progress (one square), whatever the path, change the color of the square, from white B to black N, or vice versa.
- We label the sequence of movements, on any path, with the numbers from 1 to 64.
- Putting in correspondence the numbering with sequence B, N, B, N, B, N ....., it is seen that, each even number corresponds to the black color N. Therefore, the final displacement can never bring the tower on white.

Therefore, regardless of the path taken, the tower will never reach his goal.

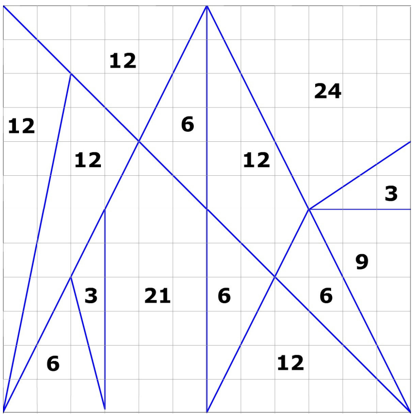
The Stomachion

On a chessboard 12 x 12 draw lines dividing the chessboard into 14 pieces, as in the following figure:



We have obtained the 14 tiles of Stomachion (see cover), an ancient game of which Archimedes studied the mathematical properties anticipating the combinatorics.

The calculation of the area of each piece of Stomachion you perform elementarily, without resorting to the Pick's theorem, obtaining the following values.

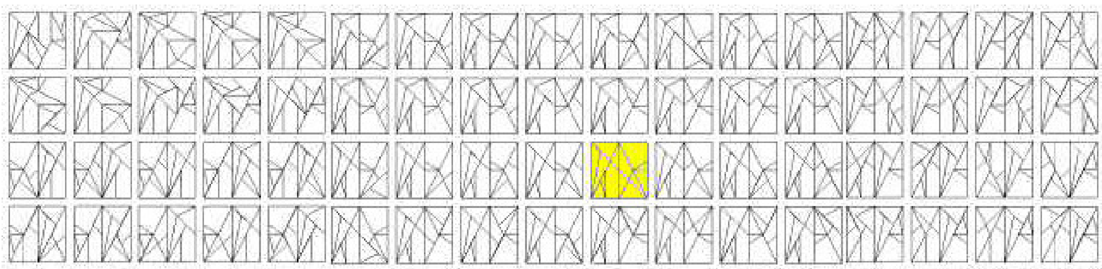


It thus appears that the 14 pieces are all commensurable with the area of the square, in the following ratios:

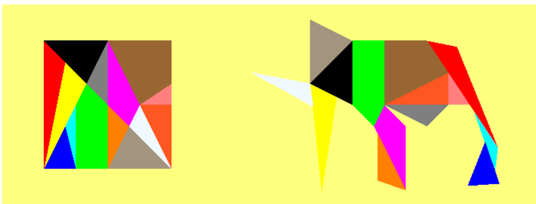
2 pieces of area 3	1/48
4 pieces of area 6	1/24
1 piece of area 9	1/16
5 pieces of area 12	1/12
1 piece of area 21	7/48
1 piece of area 24	1/6

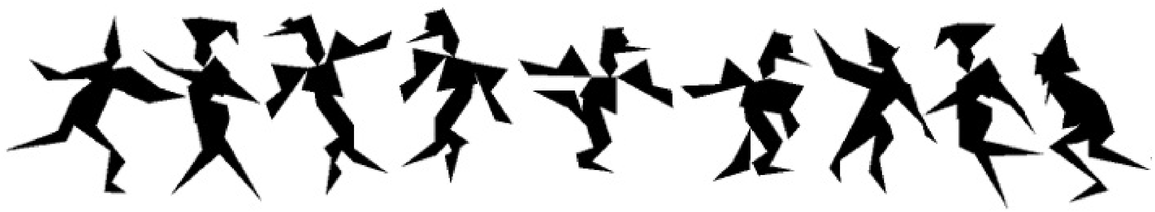
and the area of each piece is of the area of multiple smaller piece (1:48).
This property of the pieces allows to construct, using all the pieces or part of them, various geometric shapes whose areas are between them in certain ratios.

Another feature of the various polygonal shapes of the game is the existence in them of different pairs of equal sides and special relations between the various angles, for which there are many possibilities of assembly of parts in the form of a square (1). Stomachion therefore proposes a computational geometry problem, solved only recently.
A group of researchers at Stanford University has calculated that the total number of configurations obtainable in the square 12 x 12, that is, all possible solutions to the puzzle, is 17152.
Bill Cutler at Cornell University, has shown that these can be reduced to 536 distinct solutions, if we consider equivalent all configurations obtainable from each other by rotation or reflection (2).
We show below some of the 536 distinct configurations found by Cutler:



If we free the game from the requirement to assemble the separate cards in the form of the initial square, then the number of figures obtainable, using all or part of the tiles, becomes *very high*, so as to make significant the possibility to realize figures having a note physiognomy, which is the main goal of the game. In the following figures we show the *Ausonius' elephant* and others interesting figures:





<u>Notes</u>

1 - Looking Cutler's configurations you can see various ways in which you can assemble multiple pieces with sides whose sum gives the side of the square, so to be placed on one side, or pieces with angles that added together give a right angle, to place on the vertices, and other combinations.

2 - It does not take long to calculate, starting from the 17152 *possible* solutions, the 536 *distinct* solutions.
In fact, by combining the following numbers:

4 rotations around the center of the square
2 reflections respect to the sides
2 reflections respect to the axes
2 reflections respect to the diagonals

we obtain: 4 x 2 x 2 x 2 = 32

whereby: 17152 / 32 = **536** !

ON AREAS AND PERIMETERS OF REGULAR POLYGONS

Still on the guiding theme of this book , illustrated in the introduction, I made another interesting "rediscovery" by investigating a historical subject of mathematics: the discovery of the indian mathematician Madhava of Sangamagrama, made in the 1400s, according to which the value of π is obtained from this formula:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{(-1)^n}{2n+1} + \cdots = \frac{\pi}{4}.$$

Looking for a geometric model that could explain this series, I easily "calculated" using Autocad (**) the areas values of regular polygons inscribed in a unit radius circumference, starting from the equilateral triangle, and doubling subsequently the number of sides. As is known, the value of π corresponds to the area of the polygon with infinite sides (circle). I then calculated half-perimeters of same figures, as also the sequence of these values leads to π . Results are listed in the following table:

Sides	Area	Half-perimeter
3	1,299038	2,598076
6	2,598076	3,000000
12	3,000000	3,105829
24	3,105829	3,132629
48	3,132629	3,139350
96	3,139350	3,141032

The mentioned "rediscovery" lies in the fact that the two sequences are ***perfectly coincident***, unless the offset. There was to be expected, since the two sequences are built on the same geometric model and lead to the same result, even if individually they express different quantities.It must, however, give a demonstration.

(**) Once a flat "region" is drawn, its area and perimeter are obtained with the "Inquiry" and "Region / Mass Properties" commands.

Same results are obtained starting from the square and the pentagon:

Sides	Area	Half-perimeter
4	2,000000	2,828427
8	2,828427	3,061467
16	3,061467	3,121445
32	3,121445	3,136548
64	3,136548	3,140331
128	3,140331	3,141277

Sides	Area	Half-perimeter
5	2,377641	2,938926
10	2,938926	3,090170
20	3,090170	3,128689
40	3,128689	3,138364
80	3,138364	3,140785
160	3,140785	3,141391

so seems possible an extension, including regular polygons of any number of sides. Then, generalizing to circumferences of any radius *R*, these results make it possible to assert what follows:

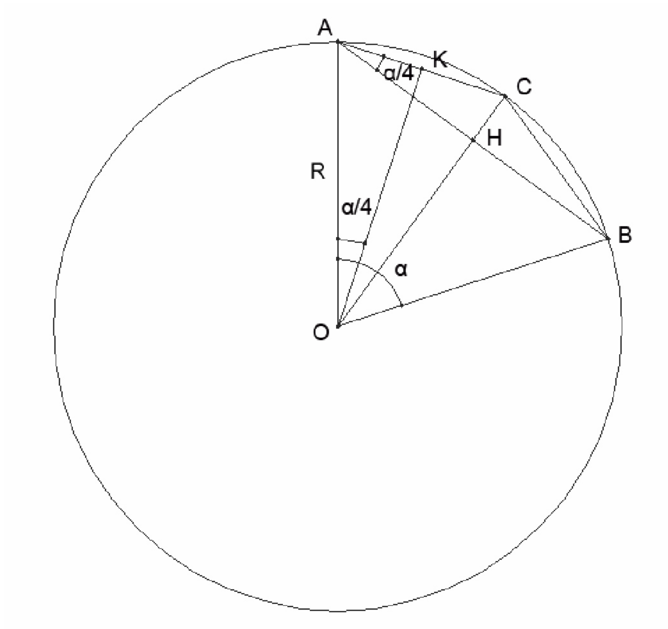
The numerical value of the half-perimeter of a regular n-sided polygon, inscribed in a circle of radius R, is equal to the numerical value of the area of the regular polygon of 2n sides inscribed in the same circle.

This statement has the value of a simple observation, without any value, since it compares heterogeneous quantities such lengths and areas. We will then give it a geometric meaning, elevating it to the rank of theorem, formulating it instead as follows:

The area of the rectangle having as its base the half-perimeter of a regular polygon of n sides inscribed in a circle of radius R, and for height the same radius R, is equal to the area of the regular polygon of 2n sides inscribed in the same circle.

Proof

Consider, in a circle of radius *R*, a generic slice of amplitude α, where α is the umpteenth part of *2* π, and we construct in it the umpteenth part of polygons of *n* and *2n* sides under consideration.

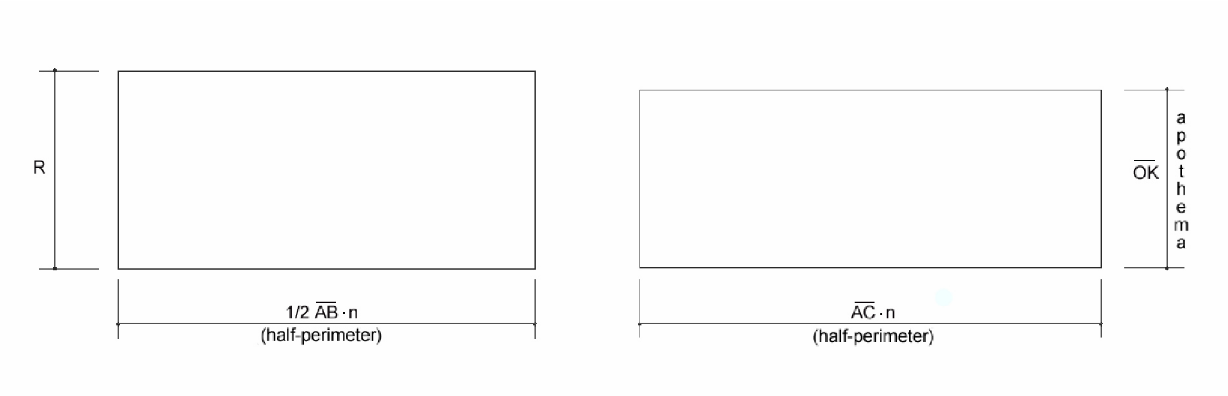


The figure shows the following relationships:

$$\overline{OK} = \overline{OA}\cos\frac{\alpha}{4} = R\cos\frac{\alpha}{4}$$

$$\overline{AC} = \frac{\overline{AH}}{\cos\frac{\alpha}{4}}$$

We must prove the equivalence of rectangles:



that is:

$$\frac{1}{2}\overline{AB} \cdot n \cdot R = \overline{AC} \cdot n \cdot \overline{OK}$$

by deleting n you have:

$$\overline{AH} \cdot R = \overline{AC} \cdot \overline{OK}$$

and replacing:

$$\overline{AC} \cdot \cos\frac{\alpha}{4} \cdot R = \overline{AC} \cdot \cos\frac{\alpha}{4} \cdot R$$

which is the identity that proves the theorem.

Remarks

Examining once again the two rectangles compared to the previous figure, it can be seen that, while the base of second rectangle "extends", with respect to the first, by a factor 1 / cos (α / 4), the height of the same "shortens" by a factor cos (α / 4). Thus, their product remains unchanged, and doubling later sides of polygons, pairs of equivalent figures are generated.

In addition, by exploiting the variable factor 1 / cos (α / 4) seen above, it seems easy, starting from the value 3 of semi-perimeter of regular hexagon, construct an infinite sequence of values whose limit is π.

Conclusion

In this article we have "dusted off" one of the many curiosities of mathematics which, due to their low importance, tend to fall into oblivion, but which never cease to amaze us, when, walking along different paths, we accidentally encounter them.

P.S.

It came to my mind, later, that one could do without trigonometry, both in the demonstration and in the observation, putting aside cos (α / 4) and exploiting instead in the reasoning, perhaps in a simpler way, the existing proportionality between sides of similar right-angled triangles OAK and ACH.